TWISTOR AND "WEAK" GAUGE STRUCTURES IN THE FRAMEWORK OF QUATERNIONIC ANALYSIS

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Abstract. The earlier proposed conditions of *(bi)quaternionic differentiability* are nonlinear, give rize to the 2-spinor and the self-dual gauge structures and may be considered as the *generating system of equations* (GSE) with respect to the source-free Maxwell, Yang-Mills and eikonal equations. We present the general solution of the GSE in terms of twistor variables, analize its rather specific gauge symmetry and demonstrate the relation of GSE to the equations of shear-free null congruences and, consequently, - to effective metrics of Kerr-Shild type. The concept of particles as singularities of physical fields associated with the solutions of GSE is developed

KEY WORDS : Noncommutative analysis; quaternionic differentiability; twistors; self-duality; shear-free congruences; Kerr-Schild metrics; singularities.

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1 Introduction

We present here an effectively nonlinear, non-Lagrangian system of PDE which follow from an invariant differential-form condition

$$d\xi = \Phi dX \xi \tag{1}$$

for a 2-spinor $\xi(x)$ and a $Mat(2,\mathbb{C})$-valued gauge $\Phi(x)$ fields. Condition (1) has been proposed in [1, 2] in the general framework of *algebrodynamical* approach to field theory [1]. In (1) d is an operator of external differentiation, and by X a hermitian 2×2 matrix of the Minkowski space-time coordinates is denoted. System (1) originates as a particular case of the *differentiability conditions* for functions of *biquaternionic* ($\mathbb{B}$-) variable (we recall that the algebra of complex quaternions $\mathbb{B}$ is isomorphic to the full algebra of 2×2 complex matrices, $\mathbb{B}\cong Mat(2,\mathbb{C})$). Geometrically, it defines a spinor field covariantly constant with respect to an effective $\mathbb{B}$-valued connection 1-form $\Omega = \Phi dX$ [2].

Remarkably, system (1) is overdetermined, so that both the 2-spinor and the gauge fields are to be found from it in a self-consistent manner. The integrability conditions for (1) take the form

$$dd\xi \equiv 0 = (d\Phi - \Phi dX\Phi)\wedge dX\xi \equiv R\xi \tag{2}$$

[1] In algebrodynamics one claims to derive field equations from only the intrinsic properties of fundamental mathematical structures, in particular from exeptional quaternion-type algebras.

[2] In a 4-vector representation this connection 1-form gives rize to an effective Weyl-Cartan geometry with the Weyl nonmetricity vector and the pseudotrace of the torsion tensor being proportional to each other, see section 5.

and, as shown below, lead to self-duality of the curvature 2-form R of the connection Ω. Consequently, the source-free Maxwell and Yang-Mills equations are satisfied identically on the solutions to (1), for the trace and the trace-free parts of the curvature R respectively. For this reason, in what follows system (1) is referred to as the *generating system of equations* (GSE).

Moreover, the 4-eikonal and the linear wave equation do hold both for each spinor component $\{\xi_A\}$, $A = 0,1$ (the latter only for the quotient of components). Also, a null geodesic congruence which may be constructed via its tangent 4-vector $k_\mu = \xi_A \xi_{A'}$ turns out to be *shear-free* on the solutions of (1) and, therefore, the *Kerr theorem* [14] may be applied to the GSE. On this track, general analytical solution to (1) may be obtained in terms of *twistor variables* $T = \{\xi, iX\xi\}$ and in an implicit algebraic form

$$\Pi^{(C)}(\xi, iX\xi) = 0, \qquad (3)$$

where $\{\Pi^{(C)}\}$, $C = 0,1$ are two arbitrary and independent holomorphic functions of 4 twistor components. If some $\Pi^{(C)}$ are chosen, a solution $\xi_A(x)$ can be extracted from the system of two algebraic equations (3), and all the analytical solutions to the GSE can be obtained in this way.

Expression (3) generalizes the Kerr theorem which deals with only one homogeneous function Π of 3 *projective* twistor components. From the full twistor structure of general solution (3) gauge symmetry of the GSE (1) of a rather peculiar type does follow, the so called "weak" one, when the gauge parameter is allowed to depend on coordinates only implicitly, i.e. via the spinor $\xi_{(0)}$ being transformed and its twistor counterpart $X\xi_{(0)}$.

Now, in a usual way a Riemann metric g of the Kerr-Shild type may be defined through the 1-form of the congruence $k = k_\mu dx^\mu$ as

$$g = \eta + h\; k \otimes k, \qquad (4)$$

with η being the flat Minkowski metric. Well known is the fact that for a shear-free congruence the metric (4) often satisfies the Einstein or Einstein-Maxwell (electro)vacuum equations under an appropriate choice of the "potential" function $h(x)$ [33].

Thus, at least the source-free Maxwell, Yang-Mills and the metric (effective gravitational) fields can be naturally assosiated with the solutions of the GSE (1). Wonderfully, singularities of strengths (of curvatures) of all these fields coincide in space and time, being determined by a single condition

$$\det \| \frac{d\Pi^{(C)}}{d\xi_A} \| = 0, \qquad (5)$$

which specifies the points where the solution to (3) becomes not unique. Geometrically, these points constitute a *caustic* of a correspondent shear-free light bundle. We'll see that the singular set, as defined by (5), may be point-like, string-like and even a two-dimensional surface. In case when this set is bounded in 3-space we can identify it with a *particle-like* object whose time evolution is fully governed by the field equations (1) and may be obtained from the algebraic system (3),(5).

These singular objects manifest, at a purely classical level, some properties of the real quantum particles; in particular, the value of electric charge q is either zero or a whole multiple of the charge of the fundamental static solution; the latter corresponds just to the Kerr shear-free congruence with the ring-like singularity. The assosiated metric and electromagnetic field are identical to those of the Kerr-Newman solution in GTR, exept for the restrictions on the admissible value of electric charge [3] $q_0 = \pm 1/4$ so that the quantity q_0 can be naturally identified with the *elementary charge*. In account of the well known property of the Kerr-Newman ansatz to have anomalous Dirac value of the *gyromagnetic ratio* [36] we conclude that the fundamental static solution to the GSE reproduces all of the quantum numbers of the real electron.

[3] In the dimensionless units we use; the numerical value itself is of no significance.

Throughout the whole paper, we make no assumptions of either physical or mathematical character. Nor we introduce some additional dynamical equations for the fields involved, or use some Lagrangian structure. What we actually do is we study the intrinsic mathematical properties of the "generalized Cauchy-Riemann equations" as they are (precisely, a particular subclass of their solutions, represented by GSE (1)). Attempts to use such structures in theoretical physics have been repeatedly undertaken, especially in the works of F.Gürsey and his co-workers [3, 4] where the self-dual gauge and the chiral structures have been reformulated on the basis of quaternionic calculus. Most of considerations, however, made use of the only recognized version of quaternionic analiticity constructed by R.Fueter [13]. Unfortunately, his approach leads to trivial linear $\mathbb{C}$-like structure of the differentiability conditions ignoring the noncommutativity of the quaternion-type algebras and considerably limiting possible physical applications of the theory.

In the paper we don't intend to discuss the problem of noncommutative analysis in detail, for this referring the reader e.g. to the profound paper of A.Sudbery [9]. In the appendix we only frmulate the conditions of $\mathbb{B}$- differentiability in the version earlier proposed in [1, 2, 24, 25] and derive the GSE (1) as a particular case of them. To our knowledge, it is the only approach in which just the nonlinear structure of the generalized CR-equations naturally emerges, as a consequence of noncommutativity of the basic algebra (the situation well known for the gauge groups but quite unfamiliar in the analysis). In view of this, one should not be surprised that the field dynamics induced by the nonlinear differentiability conditions appears to be quite nontrivial and, perhaps, has something to do with real dynamics of particles. Anyway, the main goal of this article is to develop an unexpectedly rich and self-consistent "virtual" physics which is contained in a concise and invariant structure of the $\mathbb{B}$-differentiability conditions (1) themselves, despite any additional phenomenological assumptions.

The structures which arise in this context have some features in common with the I.Robinson's theorem [5] relating shear-free congruences to (null) electromagnetic fields; with the approach of R.Penrose to the theory of spin-3/2 field relating twistor space to the charge space of electromagnetic field asssociated [7] and, especially, with spinor connections introduced by K.P.Tod [6]. From the generic viewpoint, our approach is adjacent to that of Yu.Manin and G.Henkin [8] in which the nonlinear Dirac-Maxwell system is reduced to the linear Cauchy-Riemann equations in twistor space. On the whole, however, our approach does not appeal to any others and new physical concepts arising therein are not made up or artificial but quite inevitable in a consecutive mathematical scheme used.

Let us sketch out the organization of the article. In the second section we establish the relativistic invariance of the GSE and obtain the 4-eikonal equation for the components of the 2-spinor. In section 3 we demonstrate the functional dependence of the components of a twistor assosiated with the GSE and thus reduce the latter to the system of *algebraic* equations (3). The fourth section is devoted to the analysis of the gauge symmetry of the GSE and to its specific relation to the projective transformations in the associated twistor space. In section 5 we study the self-dual structure of the GSE which follows from its integrability conditions and guarantees that the source-free Maxwell and Yang-Mills type equations are satisfied on the solutions of the GSE. In section 6 we establish the relations of GSE to the shear-free geodesic congruences and to the Riemann metrics of the Kerr-Schild type. The review of the "particle-like" solutions of GSE and the property of electric charge quantization are presented in section 7. In the last section 8 we conclude by the discussion of general problems and perspectives of the theory. In the appendix we formulate the conditions of differentiability for $\mathbb{A}$-valued functions of a noncommutative algebraic variable $Z \in \mathbb{A}$ and, for $\mathbb{A} \cong \mathbb{B}$, derive the GSE(1) as a particular case of these conditions.

Throughout the paper (exept in the appendix) the standard two-component spinor formalism is used. Upper case Latin indices range and sum over zero and one, and are raised and lowered by the symplectic spinors ϵ^{AB} and ϵ_{AB} respectively, as in [14]. By $\nabla_{AA'}$ the usual spinor derivative operator in Minkowski space is denoted.

2 Spinor structure and the eikonal equation

In the matrix representation of the biquaternion algebra $\mathbb{B}$ used in (1) we regard ξ as a column (2×1 matrix over $\mathbb{C}$), Φ – as a 2×2 complex matrix of general type, and X – as a hermitian matrix representing the coordinates of Minkowski space-time. System (1) is evidently invariant under the global *Lorentz transformations* of coordinates

$$X \to LXL^+, \quad \xi \to \bar{L}^+\xi, \quad \Phi \to \bar{L}^+\Phi\bar{L}, \quad \bar{\Phi} \to L\bar{\Phi}L^+, \tag{6}$$

where $L \in SL(2,\mathbb{C})$ and $\bar{\Phi}$ denotes the matrix *adjoint* to Φ: $\bar{\Phi}\Phi = (\det\ \Phi)$ (we admit here the possibility for $\det\Phi$ to be zero). We see that the field $\xi(x)$ transforms as a (conjugate) 2-spinor whereas the components of the field $\bar{\Phi}(x)$ constitute a complex Lorentz 4-vector (we'll identify it later with the electromagnetic 4-potential).

In view of the form of transformations (6), functions ξ,Φ are mapped respectively to matrices $\xi_{A'}$ and $\Phi_{B'A}$, whereas dX – to a hermitian matrix $dX^{AA'}$, with (un)primed indices $A,...A',... = 0,1;0',1'$ having usual 2-spinor sense. In matrix notation, (1) has then the form

$$d\xi_{B'} = \Phi_{B'A}dX^{AA'}\xi_{A'}, \tag{7}$$

equivalent to a system of eight PDE

$$\nabla_{AA'}\xi_{B'} = \Phi_{B'A}\xi_{A'}. \tag{8}$$

Throughout the paper, we assume for both spinor components $\xi_{A'}$ to be nonzero in the region of spacetime considered (otherwise, the solution to (1) can be proved to be degenerate in some sense, with zero electromagnetic and other fields associated). We also consider all the functions $\{\xi_{A'}(x),\ \Phi_{B'A}(x)\}$ to be analytical everywhere exept maybe at some subset of zero measure where they are allowed to turn to infinity.

Some properties of the solution $\xi(x)$ can be inferred directly from (8). Using the orthogonality identity $\xi^{A'}\xi_{A'} = 0$ one easily find

$$\nabla^{AA'}\xi_{C'}\nabla_{AA'}\xi_{B'} = 0, \tag{9}$$

which in particular implies the *eikonal* equation for any function $\lambda(\xi_{A'})$ of spinor components

$$\nabla^{AA'}\lambda(\xi_{A'})\nabla_{AA'}\lambda(\xi_{A'}) = 0. \tag{10}$$

Returning to system (8), multiplying it by $\xi^{A'}$ and again taking into account $\xi^{A'}\xi_{A'} = 0$ we reduce it to the following form [27]:

$$\xi^{A'}\nabla_{AA'}\xi_{B'} = 0, \tag{11}$$

where the 4-vector field $\Phi_{B'A}$ has been dispensed with. The latter may be recovered by using (8) with equal indices $A' = B'$ [4]:

$$\Phi_{A'A} = \nabla_{AA'}\ln\xi_{A'}; \tag{12}$$

certainly, no summation over A' is assumed in (12).

Conversely, from (11) in view of a well-known property of 2-spinors it follows

$$\nabla_{AA'}\xi_{B'} = \varphi_{B'A}\xi_{A'}, \tag{13}$$

with some spintensor $\varphi_{B'A}$. In compare of (13) and (8) we conclude with the equivalence of (11) to the original spinor system (8) and, therefore, – to the GSE (1).

[4] We recall that both components of the spinor considered are assumed to be nonzero: otherwise all of the strengths (curvatures) vanish identically.

3 Twistor structure and general solution of GSE

We turn now to the solutions of (11) or of GSE (8) equivalently. Remarkably, they happen to be completely determined by a *twistor structure* which can be naturally assosiated with the system considered. To demonstrate this, let us introduce another 2-spinor τ^A related to $\xi_{A'}$ via the *Klein-Penrose correspondence*

$$\tau^A = X^{AA'}\xi_{A'}. \tag{14}$$

Then a pair of 2-spinors $T^a = (\xi_{A'}, \tau^A)$, $a = 0, 1, 0', 1'$, constitute a (null) twistor *incident* with a (real) Minkowski space-time point represented by $X^{AA'}$. [5]

According to definition (14) the wedge product of the differentials $d\xi_{A'}$ and $d\tau^A$ may be formed as

$$d\tau^A \wedge d\xi_{B'} = X^{AA'} d\xi_{A'} \wedge d\xi_{B'} + \xi_{A'} dX^{AA'} \wedge d\xi_{B'}. \tag{15}$$

From (15) a rather obvious property about (nontrivial) twistors immediately follows: *at least* some two components of a generic twistor T^a should be functionally independent (as functions of coordinates $X^{AA'}$). Indeed, assuming conversely for all exterior products of the differentials $d\xi_{A'}$ and $d\tau^A$ to vanish due to their functional dependence, we get from (15): either both $\xi_{A'} \equiv 0$ or $\nabla_{AA'}\xi_{B'} = 0$, but in the second case both $\xi_{A'}$ are constant and the two components of the spinor τ^A (14) are then evidently independent.

If we subject now the spinor $\xi_{A'}$ to the dynamical system (11) the remaining two components of the twistor *should depend* on the first independent two. Precisely, we intend to prove the following:

Preposition.1 *Iff $\xi_{A'}$ is a solution of (11) then the corresponding twistor T^a has two and only two functionally independent components.*

Going to differentials in (14) and using (7), we come to equations for $d\tau^A$, which together with (7) themselves constitute a system of four equations

$$d\xi_{B'} = \Phi_{B'A} w^A, \tag{16}$$

$$d\tau^B = X^{BB'}\Phi_{B'A} w^A + w^B, \tag{17}$$

where the 1-forms $w^A = dX^{AA'}\xi_{A'}$ have been introduced. Since the differentials of twistor components are linear functions of the two 1-forms w^A only, it becomes obvious that the exterior product of any three is zero, resulting in the desired functional dependence.

The same conclusion could be reached in a slightly different way. Since two twistor components are certainly functionally independent, we can always find two equations in (16),(17) which allow to express w^A through the differentials of the corresponding twistor components. Substituting the resulting expressions in remaining two equations, we end up with two relations each containing three of the four differentials $d\xi_{A'}$ and $d\tau^A$. These relations imply the sought-for functional dependence between any three twistor components, a fact that can be expressed in a more symmetric form [27]

$$\Pi^{(C)}(T^a) \equiv \Pi^{(C)}(\xi_{A'}, \tau^B) = 0, \tag{18}$$

where $\{\Pi^{(C)}\}$, $C = 0, 1$ are two arbitrary but independent holomorphic functions of four complex variables.

Conversely, the algebraic system (18) implicitly determines $\xi_{A'}$ and it is easy to check, by differentiating (18) and multiplying by $\xi^{A'}$, that $\xi_{B'}$ satisfies the system

$$\frac{d\Pi^{(C)}}{d\xi_{B'}}\xi^{A'}\nabla_{AA'}\xi_{B'} == 0, \tag{19}$$

[5] For our purposes we may ignore imaginary unit i in definition (14).

which is equivalent to (11) except at some singular points (see below). Successively resolving system (18) at each space-time point $X^{AA'}$ with respect to $\xi_{A'}$ and substituting the resulting solution in (12) to find the corresponding "potentials" $\Phi_{B'A}$ we obtain a solution to the GSE starting from the algebraic constrains (18). This furnishes the proof of preposition.1. ■

Thus, algebraic system of equations (18) implicitly determines the *general (analytical) solution* $\{\xi_{A'}, \Phi_{B'A}\}$ of the GSE. Points where the equations (18) have multiple roots, i.e. cannot be *in a unique way* resolved for $\xi_{A'}$ satisfy according to (19) the equation

$$\det \| \frac{d\Pi^{(C)}}{d\xi_{A'}} \| = 0. \tag{20}$$

These points constitute a singular set for electromagnetic field which in the next section will be assosiated with the quantities $\Phi_{A'A}$. Together with (18) the last equation allows us to determine the shape and the time evolution of singularities (see below, section 7).

Geometrically, the algebraic system (18) defines a two-dimensional complex surface in the twistor space $\mathbb{C}^4$ (precisely, in the subspace of null twistors). Points of intersection of this surface with two-dimensional planes formed by (null) twistors (14) represent the solution $\xi_{A'}$ to the GSE (*multivalued* in general) for each fixed space-time point $X^{AA'}$. Singularities are then the pre-images (in M) of the points of twistor space at which the planes (14) are *tangent* to the surfaces (18), so that the singular set will be the same for all of the branches of a multivalued solution. Note that we ignore here the generally considered projective structure of the twistors which, in the framework of this approach, has some peculiarities and is related to an exotic gauge symmetry of GSE. These issues are discussed in the next section.

4 Projective transformations of twistors and "weak" gauge symmetry of the GSE

For an appropriate electrodynamical interpretation we need to establish the gauge invariance of the GSE, which will be dealt with in this section. Specifically, we shall study the symmetry of (8) under transformations

$$\xi_{A'} \to \xi'_{A'} = \alpha(x)\xi_{A'}, \tag{21}$$

where $\alpha(x)$ is a smooth complex function of coordinates. Using equations (11), it's readily seen that α cannot be an arbitrary function of coordinates, it rather satisfies the equation

$$\xi^{A'}\nabla_{AA'}\alpha = 0, \tag{22}$$

from which in view of 2-spinors' properties follows

$$\nabla_{AA'}\alpha = \rho_A\xi_{A'} \tag{23}$$

for some ρ_A and, consequently, – the eikonal equation for $\alpha(x)$

$$\nabla^{AA'}\alpha\nabla_{AA'}\alpha = 0. \tag{24}$$

Before we carry on, we need to establish an auxiliary result. Let us rewrite equation (17) in partial derivatives

$$\nabla_{AA'}\tau^B = X^{BB'}\Phi_{B'A}\xi_{A'} + \xi_{A'}\delta^B_A. \tag{25}$$

Using the orthogonality $\xi^{A'}\xi_{A'} = 0$, we immediately verify the validity of the following relations:

$$\nabla^{AA'}\xi^{B'}\nabla_{AA'}\tau^{B} = 0, \tag{26}$$

$$\nabla^{AA'}\tau^{B}\nabla_{AA'}\tau^{C} = 0, \tag{27}$$

which along with equation (9) lead to the eikonal equation for any function $\lambda(T^a)$ of twistor components

$$\nabla^{AA'}\lambda(T^a)\nabla_{AA'}\lambda(T^a) = 0. \tag{28}$$

Going now back to the main goal of this section and taking into consideration the eikonal equation for $\alpha(x)$, we are guided by the equation (28) to conjecture the following

Preposition.2 *Transformation of the type (21) are symmetries of (8) iff α is a function of T^a and $\Phi_{B'A}$ transforms according to*

$$\Phi_{B'A} \to \Phi'_{B'A} = \Phi_{B'A} + \nabla_{AB'} \ln\alpha. \tag{29}$$

Replacing in (8) $\xi_{A'}$ and $\Phi_{B'A}$ by their transformed values, after some simplification we obtain the following condition of form-invariance of (8):

$$\xi_{B'}\nabla_{AA'}\alpha - \xi_{A'}\nabla_{AB'}\alpha = 0, \tag{30}$$

which is skew-symmetric in A', B' and therefore equivalent to the equation (22) for $\alpha(x)$. Taking now into account equations (8) and (25) and carrying out simple manipulations we show that if $\alpha = \alpha(T^a) = \alpha(\xi_{A'}, \tau^B)$ then equation (22) is identically satisfied. This proves the sufficient part of the preposition.

To prove the converse, suppose that transformation (21) is a symmetry of (8). This yields the following:

$$\xi_{B'}d\alpha = \alpha(\Phi'_{B'A} - \Phi_{B'A})w^A, \tag{31}$$

where, as before, $w^A = dX^{AA'}\xi_{A'}$. Making use of (16) and (17), it's easy to see that the exterior product of equation (31) with any two differentials of the twistor components vanishes, leading to the functional dependence of α on the corresponding twistor components. More symmetrically, this result can be expressed as asserted in the preposition, i.e. $\alpha = \alpha(T^a)$. Passing then to partial derivatives in (31) we obtain the relation

$$\xi_{B'}\nabla_{AA'} \ln\alpha = (\Phi'_{B'A} - \Phi_{B'A})\xi_{A'}\ , \tag{32}$$

from which the desired transformation rule (29) for the "potentials" $\Phi_{B'A}$ follows (in a special case when the primed subscripts are equal). This completes the proof of preposition.2. ■

Some words are in order about the nature of transformations

$$\xi \to \xi' = \alpha(T^a)\xi. \tag{33}$$

which may be called *restricted*, or *weak* gauge transformations [28]. We remark that according to its definition (14) the conjugate spinor τ^B transforms in a similar way

$$\tau \to \tau' = \alpha(T^a)\tau \tag{34}$$

so that both (33),(34) together imply that the gauge symmetries of GSE may be considered as the transformations of twistors of the form

$$T \to T' = \alpha(T^a)T. \tag{35}$$

It's clear that composition of transformations (35) is a transformation of the same type. Their associativity is also obvious. But the existence of inverse transformation is not so evident. However, according to preposition.1, T^a and its image T'^a both have only two functionally independent components, and $\alpha(T^a)$ depends, essentially, on these two components (of T^a). So we can always express the two independent components of T^a through those of T'^a. Substituting them in $\alpha^{-1}(T^a)$ results in the inverse transformation $T = \alpha^{-1}(T'^a)T'$ which is of the same type as (35). Hence *these transformations constitute a group*! In fact it is a proper subgroup of the full $\mathbb{C}(1)$-gauge group of transformations (21), the latter itself being generally not a symmetry of the GSE . The statement that this subgroup is a proper one, becomes quite evident if we recall that $\alpha(T^a)$ should be subject to the eikonal equation (24).

Finally, we note that under the transformations (33) the *trace* part of the matrix 1-form $A \equiv Tr(\Phi dz) = \Phi_{A'A}dX^{AA'}$ with components $\bar{\Phi}_{AA'} \equiv \Phi_{A'A}$ transforms gradient-wise

$$A \rightarrow A + d\ln\alpha, \tag{36}$$

as the electromagnetic potential 1-form does under the gauge transformations (this may be also seen from the expression (12)). In view of the 4-vector properties of $\bar{\Phi}$ under Lorentz transformations (6) we are brought to adopt the interpretation of the 1-form A as the potentials and of the respective gauge-invariant 2-form $F = dA$ as the electromagnetic field strengths (of course, up to an arbitrary scale factor only). In the following section we obtain Maxwell equations for this 2-form, therefore elaborating its electromagnetic interpretation.

Let us now look at the gauge transformations (35) from the viewpoint of the geometry of twistor space. The Abelian nature of the transformations studied results in the fact that the *ratio* of any two twistor components T^a is invariant under them. Thus, such transformations are *projective* in origin. Not only the planes (14) but also the surfaces (18) are form-invariant under transformations (35) and, consequently, give rise to another solution of the GSE (with the same electromagnetic 2-form F). So we may consider the *equivalevce classes* of the solutions (and of the surfaces (18) respectively) which can be obtained one from another via the gauge transformations (35). That's why we may restrict ourselves to consider only the *projective twistor space* CP^3. However, projective structure of this type differs essentially from the conventional one which originates from the transformations of the full gauge group (21). We shall return to this problem in section 6, and for the time being shall deal with the full structure of the space of (null) twistors.

5 Integrability conditions, self-duality and the source-free gauge equations

As was mentioned in the introduction, the GSE (1) may be viewed at as the condition that must be met for a spinor $\xi(x)$ to be covariantly constant with respect to the $\mathbb{B}$-connection 1-form

$$\Omega = \Phi dz. \tag{37}$$

It may be interesting to note that in the 4-vector representation $\mathbb{B}$-connection (37) turns into the affine connection of the form [1, 2]

$$\Gamma^{\mu}_{\nu\rho} = \delta^{\mu}_{\nu}\Phi_{\rho} + \delta^{\mu}_{\rho}\Phi_{\nu} - \eta_{\rho\nu}\Phi^{\mu} - i\epsilon^{\mu}_{.\nu\rho\lambda}\Phi^{\lambda} \tag{38}$$

which gives rise to the effective complexified geometry of *Weyl-Cartan type.* For this $\mathbb{B}$-induced geometry the Weyl vector of nonmetricity and the pseudotrace of the torsion tensor appear to be proportional to each other and are both expressed via the components of the basic gauge field $\Phi(x)$ [6].

According to definition (37), the initial GSE (1) may be rewritten as follows:

$$d\xi = \Omega\xi \tag{39}$$

The GSE is overdetermined (8 equations for 6 unknown functions) and both the spinor and the "connection" gauge fields are to be determined from it. Dynamics of the connection field $\Omega(x)$ can be obtained by external differentiation of (39) which yields

$$R\xi \equiv (d\Omega - \Omega\wedge\Omega)\xi = 0, \tag{40}$$

where in parentheses the matrix *curvature 2-form* R of the connection (37) arises. Since the spinor ξ is not arbitrary but subject to (39) the *integrability conditions* (40) don't imply the triviality of curvature [7], instead they lead to its *self-duality* [1, 2].

To demonstrate this, we note that for connection (37) the curvature R is of the following, rather specific form

$$R = (d\Phi - \Phi dz\Phi)\wedge dz \equiv \pi\wedge dz, \tag{41}$$

where a new $\mathbb{B}$-valued 1-form π emerges, with the components

$$\pi_{A'C} = \pi_{A'CBB'}dX^{BB'} = (\nabla_{BB'}\Phi_{A'C} - \Phi_{A'B}\Phi_{B'C})dX^{BB'}. \tag{42}$$

The integrability conditions (40) take then the form $(\pi\wedge dz)\xi = 0$ or, in matrix notation

$$\pi_{A'CBB'}dX^{BB'}\wedge dX^{CC'}\xi_{C'} = 0. \tag{43}$$

Making use of symmetry properties we obtain from the last relation

$$\pi_{A'}{}^{C}{}_{C(B'}\xi_{C')} = 0, \tag{44}$$

so that for any nontrivial solution $\xi(x)$ it follows

$$\pi_{A'}{}^{C}{}_{CB'} \equiv \nabla_{CB'}\Phi_{A'}{}^{C} + \Phi_{B'C}\Phi_{A'}{}^{C} = 0. \tag{45}$$

Decomposing now in a usual way the curvature (41) into the self- and antiself-dual parts it is easy to verify that equations (45) are just the conditions for its self-dual part to vanish, whereas the other (antiself-dual) one $\bar{R}$ takes the form

$$\bar{R}_{A'}{}^{C'}{}_{(BC)} = \nabla^{C'}{}_{(B}\Phi_{A'C)} - \Phi^{C'}{}_{(B}\Phi_{A'C)} \tag{46}$$

and satisfy additional integrability conditions $\bar{R}\xi = 0$ (we'll not make use of them below).

Thus, though the curvature 2-form (41) of the connection 1-form (37) is not identically (anti)self-dual (i.e. self-dual in the "strong" sense), it nesessarily becomes (anti)self-dual *on the solutions of the GSE.* For this reason, the property has been called in [27] *weak self-duality.*

[6] Vector fields covariantly constant with respect to the torsion-free Weyl connection have been studied in [16]; they are closely related to the symmetries of Weyl manifolds [17]. Relations between the nonmetrical and torsion parts of such connections were considered in [6].

[7] At this point our approach differs essentially from that of Buchdahl [18], Penrose [19] and Plebanski [20] who assumed that the integrability conditions like (40) should be satisfied identically for an arbitrary spinor field, in order to ensure the existence of an "exact set" of solutions to field equations.

Physically, the expression (46) represents the strength of a matrix gauge field; in particular, its trace part $F_{BC} = \bar{R}_{A'}{}^{A'}{}_{(BC)} = \nabla^{A'}{}_{(B}\Phi_{A'C)}$ corresponds to the aforedefined electromagnetic field strength $F = dA$ whereas the trace-free part of (46) defines the strength of a *Yang-Mills type' field* [8]

Indeed, in view of *Bianchi identities*

$$dR \equiv \Omega \wedge R - R \wedge \Omega, \tag{47}$$

self-duality of curvature $R + iR^* = 0$ leads to the source-free Maxwell equations for the electromagnetic 2-form $F = Tr(R) = R_{A'}{}^{A'}$

$$dF^* = 0 = dF \equiv 0, \tag{48}$$

and to the equations of Yang-Mills type for the trace-free part of curvature $\mathbf{F}_{A'}{}^{B'} = R_{A'}{}^{B'} - \frac{1}{2}F\delta_{A'}^{B'}$.

Generally speaking, electromagnetic 2-form F is a $\mathbb{C}$-valued field, yet in view of its self-duality it reduces to an $\mathbb{R}$-valued 2-form $\mathtt{F}$ related to F by

$$F = \mathtt{F} - \mathtt{i}\mathtt{F}^*, \tag{49}$$

for which Maxwell equations do hold as well, so that the number of its degrees of freedom is just equal to that of a usual real electromagnetic field. Explicitly, from the symmetric part of the integrability conditions (45) we get for the $\mathbb{C}$-valued "electric" $\vec{E}$ and "magnetic" $\vec{H}$ field strengths

$$\vec{E} + i\vec{H} = 0, \tag{50}$$

so that $\Im(\vec{H}) = \Re(\vec{E})$, $\Im(\vec{E}) = -\Re(\vec{H})$ and the pair $\{\Re(\vec{E}), \Re(\vec{H})\}$ represents an $\mathbb{R}$-valued electromagnetic field subject to Maxwell equations. Note that from the skew-symmetric part of (45) in the 4-vector form we get the following "inhomogeneous Lorentz condition" [1, 2] for the $\mathbb{C}$-valued electromagnetic potentials $A_\mu = \Phi_{A'A}$:

$$\partial_\mu A^\mu + 2A_\mu A^\mu = 0, \tag{51}$$

which is also identically satisfied on each solution to the GSE. Condition (51) is by no means gauge invariant in a usual sense but it *is* invariant with respect to the weak gauge transformations (33) provided the potentials satisfy the GSE.

As to the fields of Yang-Mills type, they may be always expressed via the electromagnetic field strengths and the spinor $\xi_{A'}$ itself and for this reason can't be regarded independently. Note also that the real or imaginary part of the trace-free curvature $\mathbf{F}_{A'}{}^{B'}$ being taken separately would not satisfy the source-free YM equations, in view of nonlinearity of the latters. Therefore, the YM-like fields here *are necessarily complex.* Other details about the pecularities of YM fields in the framework of algebrodynamical approach can be found in [2].

6 GSE and the null shear-free geodesic congruences

Let us recall that via the elimination of potentials $\Phi_{A'A}$ the GSE (1) takes the form (11). Once its solution $\xi_{A'}(x)$ is found, a field of a null 4-vector $k_\mu(x)$, $k_\mu k^\mu = 0$ can be defined as

$$k = k_\mu dx^\mu \equiv \xi_A \xi_{A'} dX^{AA'}. \tag{52}$$

[8] Owing to the restricted (weak) gauge symmetry this is not precisely what is usually understood as the YM fields; however, the dynamical structure of the gauge equations is completely the same, the restrictions are imposed only on the solutions.

Vector lines of this field define a null congruence for which the shear-free criterion [14]

$$\xi^{A'}\xi^{B'}\nabla_{AA'}\xi_{B'} = 0 \tag{53}$$

follows readily from (11). Hence each solution $\xi_{A'}$ of the GSE actually defines a *null shear-free geodesic congruence* (SFC). Contrary to (11), the SFC equations (53) are invariant under the full complex Abelian gauge group (21) and reduce to the system of two equations in partial derivatives

$$\nabla_w G = G\nabla_u G, \quad \nabla_v G = G\nabla_{\bar{w}} G, \tag{54}$$

where $G = \xi_{1'}/\xi_{0'}$ is the gauge invariant, and the following generally accepted notation has been used:

$$X^{AA'} = \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix} \equiv \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{55}$$

Four real quantities $\{x^i; x^0\}$, $i = 1, 2, 3$ correspond to the Cartesian space and time coordinates respectively. Note that as for the individual spinor components $\xi_{A'}$, they remain fully indeterminate by the SFC equations which impose restrictions only on their quotient $G(x)$.

Let us compare this with the GSE (11). The latter is equivalent to the system of *four* equations for only two spinor compoments $\xi_{A'}$

$$\nabla_w \xi_{A'} = G\nabla_u \xi_{A'}, \quad \nabla_v \xi_{A'} = G\nabla_{\bar{w}} \xi_{A'} \tag{56}$$

from which again the equations (54) follow for the quotient $G(x)$. Multiple solutions of (56) with the same G correspond to different potentials but have the same strengths of the electromagnetic and YM fields assosiated. In view of this, further on we identify the GSE and SFC equations by considering only the *projectively invariant part* of the GSE represented by system (54) (one may regard this as a fixing of the gauge $\xi_{0'} = 1$).

General analytical solution of (54) for $G(x)$ immediately follows now from the preposition.1. in the form of an algebraic equation

$$\Pi(G, \tau^0, \tau^1) = \Pi(G, u + wG, \bar{w} + vG) = 0, \tag{57}$$

which determines implicitly the function $G(x)$. Here Π is an arbitrary holomorphic function of three complex variables. Equation (57) manifests the functional dependence of three components G, τ^0, τ^1 of a projective twistor T^a assosiated with the solutions of GSE. For the SFC equations (53) the equivalent result is well known as the *Kerr theorem* [14]. Note that the solutions of (54) in the form (57) are defined exept at the points of the singular set whose equation (20) simplifies now to

$$P \equiv \frac{d\Pi}{dG} = 0. \tag{58}$$

By multiplying the two equations of (54) we obtain once more the 4-eikonal equation for $G(x)$ in the form

$$\nabla_u G \nabla_v G - \nabla_w G \nabla_{\bar{w}} G = 0, \tag{59}$$

while by differentiating them we verify that $G(x)$ satisfies also the linear d'Alembert equation

$$\Box G(x) \equiv (\nabla_u \nabla_v - \nabla_w \nabla_{\bar{w}})G(x) = 0. \tag{60}$$

Note that in view of (59) every C^2-function $\lambda(G)$ is also harmonic on the solutions of the GSE,

$$\Box \lambda(G) = 0. \tag{61}$$

Using now ansatz (12) for the potentials $\Phi_{A'A}$ and taking into account (59), we can express the electromagnetic field strengths via the 2-nd order derivatives of $\ln G$ as

$$F_{00} = \nabla_u \nabla_{\bar{w}} \ln G, \quad F_{11} = \nabla_v \nabla_w \ln G, \quad F_{01} = \nabla_w \nabla_{\bar{w}} \ln G, \tag{62}$$

so that satisfaction of the source-free Maxwell equations is then ensured for (62) in consequence of the wave equation (61) for $\lambda = \ln G$. Differentiating twice the identity (57) with respect to the (spinor) space-time coordinates we obtain finally for the strengths (62) the following symmetric expression:

$$F_{AB} = \frac{1}{P}\left(\Pi_{AB} - \frac{d}{dG}(\frac{\Pi_A \Pi_B}{P})\right), \tag{63}$$

where the function P is defined by (58) and $\{\Pi_A, \Pi_{AB}\}$, $A, B = 0, 1$ denote the (1-st and 2-nd order) partial derivatives of Π with respect to its twistor arguments τ^0, τ^1. We return to this expression below.

Close relation of the GSE to the SFC makes it possible to introduce one more geometrophysical structure – an effective *Riemann metric.* In fact, it's well-known [31, 33] that we can transform the flat Minkowski metric $\eta_{\mu\nu}$ into a metric $g_{\mu\nu}$ of the Kerr-Schild type

$$g_{\mu\nu} = \eta_{\mu\nu} + h k_\mu k_\nu, \tag{64}$$

and that the main characteristics of SFC k_μ (geodesity, twist and shear) are preserved under this transformation. Here h is a scalar field and the congruence k given by (52) acquires a projectively invariant form

$$k = du + \bar{G}dw + Gd\bar{w} + G\bar{G}dv, \tag{65}$$

$\bar{G}$ being complex conjugated of G. Now we resort to the results of the classical paper [31] where it has been proved that the metric (64) satisfies Einstein-Maxwell electrovacuum system for any function G obeying algebraic constraint (57), with the function Π *linear* in twistor arguments τ^0, τ^1 :

$$\Pi = \varphi + (qG + s)\tau^1 - (pG + \bar{q})\tau^0. \tag{66}$$

Here $\varphi = \varphi(G)$ is an arbitrary analytic function of the complex variable G, s and p are real constants and q is a complex constant. Not going into details we just note that according to the results of [31] the scalar field h in (64) is determined, up to an arbitrary real constant, by the function Π and some another function $\Psi(G)$ independent of $\varphi(G)$ and related to the electromagnetic field arising therein. These electromagnetic fields are defined in the curved space-time with metric (64), and generally they are different from those emerging in our approach and satisfying Maxwell equations in the *flat* space-time[9]. However, for the most fundamental Reisner-Nördstrem and Kerr-Newman solutions these fields coincide, the only difference being in that in our approach the electric charge is fixed in magnitude by the GSE itself (see the next section).

In [32, 31] it was shown that singularities of Riemann curvature of the Kerr-Schild metric (64) correspond just to the condition (58). On the other hand, the expression (63) demonstrates that the same condition $P = 0$ determines the set of singular points of the electromagnetic field. It may be verified that this is true also for the strengths of Yang-Mills fields assosiated with the solutions of the GSE.

Hence, with each solution of the GSE an electromagnetic, a $\mathbb{C}$-valued Yang-Mills and an effective gravitational field can be naturally associated, their singularities are all determined by equation (58) and therefore coincide in space and time. This makes it possible, in the framework of the algebrodynamical approach based on the GSE, *to consider particles as common singularities of all fields involved.* This general concept will be developed in the next section.

[9]At the same time they both are generally different from the fields which may be defined for the SFC using the Penrose twistor transform, see [14, 15].

7 Quantization of electric charge and "particle-like" solutions of GSE

We'll briefly review now the main solutions of the GSE known to date which all may be obtained by an appropriate choice of the function Π and solving subsequently the algebraic equation (54). In order to find the solutions in a simple explicit form one usually restricts the consideration to functions Π which are *quadratic* in G (linear functions result in zero electromagnetic strengths (62)). The fundamental *static* solution is generated by a function

$$\Pi = G\tau^0 - \tau^1 + 2ia \equiv G(u + wG) - (\bar{w} + vG) + 2ia = 0,$$

($a = Const \in \mathbb{R}$), from where we get

$$G = \frac{\bar{w}}{(z + ia) \pm r_*} \equiv \frac{x + iy}{(z + ia) \pm \sqrt{x^2 + y^2 + (z + ia)^2}}. \tag{67}$$

Electromagnetic field (62) assosiated with this solution

$$\vec{E} - i\vec{H} = \pm \frac{\vec{r}_*}{4(r_*)^{3/2}}; \qquad (\vec{E} + i\vec{H} = 0), \tag{68}$$

where $\vec{r}_* = \{x, y, z + ia\}$ has a *ring singularity* of radius a, an only possible electric charge $q = \pm 1/4$ (in the dimensionless units we use), a dipole magnetic and a quadrupole electric moments equals to qa and qa^2 respectively [28]. Apart from the restriction on charge, the electromagnetic field (68) together with the Riemann metric assosiated with (67) via the SFC (65), accurately reproduce the field and metric of the Kerr-Newman solution (in the coordinates used in [31]). Particularly, for $a = 0$ the solution (67) represents the *stereographic map* $S^2 \to \mathbb{C}$ whereas the fields turn into the Coulomb one and the Reisner-Nördstrem metric respectively.

Quantization of electric charge seems to be a profound property of the GSE-solutions discovered in [1, 2]. It is a consequence of self-duality condition (50) which together with the gauge symmetry of GSE ensures the relation $q = N/4, N \in \mathbb{Z}$ for the values of electric charge assosiated with every solution of the GSE. This property has both topological and dynamical origins, the latter being related to the overdetermined structure of the GSE. The proof of the general theorem on charge quantization will be presented elsewhere. Contrary to the recently developed [34, 35] approaches to the problem of quantization of electric charge, which are purely topological, in the framework of the GSE the charge of the fundamental static solution (67) can be of only one fixed and minimal possible value and, therefore, can be naturally identified with the *elementary charge*. Together with the well-known property of Kerr-Newman solution to fix the gyromagnetic ratio $g = 2$ equal to the ratio of the Dirac particle [36] the appearance of elementary electric charge within the theory makes it much more legitimate to interprete the fundamental solution (67) as a classical model of electron (in comparison, say, with the models of Lopez [37], Israel [38] and Burinskii [39] based on Einstein-Maxwell theory itself).

According to a general theorem proved in [32] *static* solutions of the SFC (and thus of the GSE) for which the singular set is bounded in 3-space (below we call them *particle-like* [30]) are all exhausted by the Kerr solution (67) (up to 3-translations and 3-rotations). If, however, we remove the condition for a solution to be static and get out of the class of functions (66) dealt with in [31] we discover a lot of time-dependent "particle-like" solutions with bounded singularities of different dimensions, 3-shapes and time evolution.

In particular, an *axisymmetric* solution of a pacticle-like type generated by the function

$$\Pi = \tau^0\tau^1 + b^2G^2 = 0, \quad b = Const, \tag{69}$$

has been found in [27, 29]. For real b it corresponds to the case of two singularities with elementary charges $+1/4$ and $-1/4$ undergoing head-on hyperbolic motion for which the electromagnetic field

$$E_\rho = \pm\frac{8b^2\rho z}{\Delta^{3/2}}, \quad E_z = \mp\frac{4b^2 M}{\Delta^{3/2}}, \quad H_\varphi = \pm\frac{8b^2\rho t}{\Delta^{3/2}}, \tag{70}$$

is identical to that of the *Born solution.* Here the following notation is used:

$$\rho^2 = x^2 + y^2, \quad s^2 = t^2 - z^2, \quad M = s^2 + \rho^2 + b^2, \quad \Delta = M^2 - 4s^2\rho^2,$$

and singularities are defined by the condition $\Delta = 0$. For imaginary $b = ia,\ a \in \mathbb{R}$ one has at $t = 0$ a neutral ring-like singularity which then expands to a *torus.* After an interval of time $t > |a|$ singularity turns into a *self-intersecting torus* depicted in Fig.1.

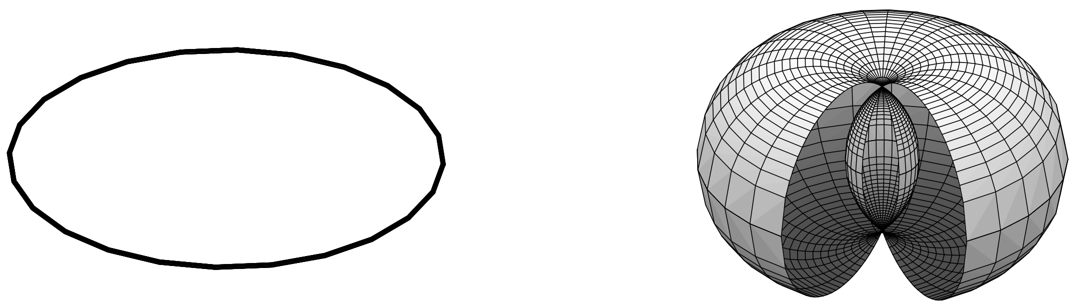

Figure 1: Singular set of electromagnetic field (70) for neutral solution (69) at initial ($t = 0$) and final ($t > |a|$) stages

We mention also a particle-like solution for which the singularity has a plane *8-figure* shape at $t = 0$, as well as a wave-type solution with a *helix-like* singularity [30]. The latter stands for the analogue of electromagnetic waves in electrodynamics based on the GSE.

A beautiful representation of the solutions of SFC-equations has been proposed by E.T.Newman [40] and developed then in the works A.Burinskii, R.P.Kerr and Z.Perjés [41]. They regarded a point-like source moving along some curve in a *complexified* Minkowski space-time $\mathbb{C}M$. Then a trace of its complex light cone on the *real* space-time M is just a null congruence which *turns out to be shear-free.* Kerr congruence is the simplest example of this representation (a point-like source resting at some place of the orthogonal to M *imaginary* subspace of $\mathbb{C}M$). The examples above-presented demonstrate, however, that the structure of singular set for such "complexified Lienard-Wiechert field" can be quite nontrivial.

To conclude, a number of exact solutions of the source-free Maxwell equations may be obtained in a purely algebraic way, some of them being unknown before. They are defined exept at the points where the electromagnetic field turn to infinity. These points constitute a set which may be 0-, 1- or 2-dimensional and for a particle-like solution is bounded in 3-space. Generally, it's impossible to cover such set by some δ-like source (because of the multivaluedness of the Kerr-type solutions). Nonetheless, the quantum numbers, 3-shape and time evolution of the singularities are well defined and nontrivial owing to the *"hidden nonlinearity"* of Maxwell equations in this theory, i.e. to their origination from the primary nonlinear GSE. The latter ensures some "selection rules" to exist for the solutions of Maxwell equation, in particular the restrictions for the allowed values of electric charge and the violation of the superposition principle (generally, the superposed solution may satisfy linear Maxwell equations but not the GSE itself). A detailed discussion of the status of singular particle-like solutions may be found in [30].

8 Conclusion

In this paper we didn't claim to present an alternative field model or an algebraic method to derive the solutions of classical field theory' conventional equations. We only attempted to study the properties of differentiable functions of $\mathbb{B}$-variable themselves, i.e. to construct a new version of noncommutative analysis. In the particular case considered, the generic conditions of $\mathbb{B}$-differentiability [1, 2, 24] reduce to the GSE (1) which naturally involves the gauge and the 2-spinor structures and manifests wonderful correlations with the structures and the language generally accepted in classical field theory.

In fact, we make only three fundamental assumptions in order to physically interpret the abstract mathematical scheme we develop:
1) about space-time as a (4-dimensional subset of the) vector space of $\mathbb{B}$-algebra,
2) about physical fields as differentiable functions of $\mathbb{B}$-variable and
3) about particles as bounded singularities of strengths (curvatures) of gauge and metric fields naturally assosiated with original $\mathbb{B}$-differentiable functions-fields.

From the physical viewpoint, the GSE may be regarded as some peculiar (nonlinear, non-Lagrangian, overdetermined) field model dealing with effectively interacting 2-spinor and electromagnetic fields, the dynamical equations for the latter being not postulated but derived from the GSE itself as its integrability conditions.

Twistor structure arises in the theory in a natural "dynamical" way via the integration of the GSE and makes it possible to obtain all its solutions, as well as correspondent solutions of gauge equations, in a fairly simple algebraic way [10].

Explicitly, by the examination of algebraic equation (57) a wide class of exact solutions, of linear Maxwell equations in particular, may be found, solutions with bounded singularities among them. Condition (20) performs the role of *equation of motion* for this particle-like objects but at the same time predetermines the charactetristics and the spatial shape of singularities realizing in such a way Einstein's hypothesis of *super-causality* [23].

Due to violation of the superposition principle for the original GSE, the evolution of the particle-like objects simulates physical interaction, and the dynamical *perestroikas* of the structure of singular sets may be interpreted as the *transmutations* of particles. All this reveals close relations to the theory of singularities of differentiable mappings and to the catastrophe theory [21]. For example, in GTR the singularity condition (20) is recognized as the condition of *caustics* of corresponding light-like beams formed by the shear-free congruences (65).

It seems also that at least some remarkable properties of the GSE may have consequences for field theory in general. In particular, we refer here to
1) a possible expansion of a class of gauge models in account of weak gauge symmetry (35) discovered and using the connections of the form (37),(38);
2) a natural possibility to obtain the selection rules for electric charge, spin etc. using the overdetermined structure of field equations;
3) a total algebraization of the primary PDE equipped by a twistor structure using the analogue of Kerr theorem or its generalizations like (18);

[10] In R.Penrose's twistor approach we need to integrate a function of twistor variable in order to get a solution to the wave equations; contrary, in our approach even this is not necessary.

4) a possibility to establish the structure and evolution of singularities without even explicitly obtaining the solution of field equations itself (via elimination of the main field function $G(x)$ from the system of algebraic equations (57) and (20), a procedure proposed in [32]).

There are at least three ways to consider the material presented in the paper and the role of GSE in particular: as a beatiful mathematical toy, as a powerful method to generate the solutions of conventional field equations, and as a fundamental dynamical system (or an example of such) primary to conventional Lagrangian systems. The utilization of classical dynamics based on the overdetermined structures like GSE requires quite new methods of quantization. Alternatively, one can claim to explain the quantum properties on the whole via, say, the stochastic behaviour of an ensemble of particle-like (singular) solutions or by other means of a consistently classical consideration.

In any case, to find the correct approach to quantization and to physical interpretation in general one needs to study accurately the properties of the classical solutions themselves [11]: their complete classification, dynamics and bifurcations. All these problems are evidently related to the general theory of singularities of differentiable maps [21, 22]. Nevertheless, the already discovered properties of $\mathbb{B}$-differentiable functions-fields and of numerous geometrophysical structures they give rise to, look like somewhat striking and bring one back to Pithagorean philosophy about the *numerical origin* of fundamental physical laws.

9 Appendix

The below reviewed approach to differentiability in the quaternion-like algebras have been motivated by the old works of G.Sheffers [11] (see also [12]) on the analysis over an arbitrary *commutative* associative algebra $\mathbb{A}$, and is a direct generalization of Sheffer's approach to the noncommutative case. Let $F(Z)$ be an $\mathbb{A}$-valued function $F : \mathbb{A} \mapsto \mathbb{A}$ of an $\mathbb{A}$-variable $Z \in \mathbb{A}$. Sheffers defines the condition of its *differentiability in* $\mathbb{A}$ using proportionality of the linear parts of increments (differentials) dZ, dF of independent variable and its function respectively as

$$dF = H(Z) * dZ, \tag{71}$$

where $H \in \mathbb{A}$ and $(*)$ denotes multuplication in $\mathbb{B}$. For division algebras (71) is equivalent to the condition of existence and path-independence of the derivative $H(Z) = dF * dZ^{-1} \equiv F'(Z)$, and in a particular case of complex algebra $\mathbb{C}$ leads to Cauchy-Riemann equations. Generally, however, (71) may be applied to the algebras with zero divisors, in particular to that of the double and dual numbers. Linear differential equations relating the components of $F(Z)$ follow from (71) via the elimination of $H(Z)$ and fully resemble the CR-equations for the functions of complex variable. In many aspects the analysis constructed by Sheffers is quite similar to the complex one, so that a wide class of $\mathbb{A}$-differentiable functions subject to (71) can be found, including all the polinoms in particular.

In the r.h.s of (71) an invariant $\mathbb{A}$-valued differential 1-form of the most general type is present, which can be constructed via the algebraic operations in $\mathbb{A}$ only. A natural generalization of (71) to the case when $\mathbb{A}$ is noncommutative (yet associative) seems to be the following condition (see [1, 2] and the references therein):

$$dF = L(Z) * dZ * R(Z) \tag{72}$$

for the function $F(Z)$ to be differentiable in $\mathbb{A}$. Here $L, R \in \mathbb{A}$ are the so called left and right *semi-derivatives* of $F(Z)$ respectively. For a given function F they are determined (if exist) not uniquely, but at least up to a transformation $L \to \alpha L,\ R \to \alpha^{-1} R$, where the function $\alpha(Z)$ takes the values in the *centre* (the commutative subalgebra) of $\mathbb{A}$.

[11] To clarify the correspondence with quantum theory the particle-like *multisingular* solutions are especially interesting.

For commutative algebras (72) reduces back to (71) with $H(Z) \equiv L(Z)*R(Z)$. On the other hand, if in noncommutative case we take, say, $R = e$ (we assume the unit element e to exist in $\mathbb{A}$), we come to the condition (71) with $H(Z) \equiv L(Z)$. Nonetheless, at least for the algebras of quaternion type this condition is known to be too restrictive, being satisfied only by the linear function $F = A * Z + B$, where A, B are constant elements of algebra (see e.g. [9, 10]).

As to the general $\mathbb{A}$-differentiability condition (72), it defines a wider class of functions. Particularly, for Hamilton quaternions $\mathbb{H}$ condition (72) turns out to be an algebraic analogue of for the mapping $Z \mapsto F(Z)$ *to be conformal* in E^4 [25, 26, 24]. In this regard equation (72) can be viewed at as a natural generalization of complex holomorphy. However, in E^4 conformal mappings constitute only a finite 15-parameter group, contrary to the infinite-dimensional complex case. Thus, for division algebra $\mathbb{H}$ the class of $\mathbb{H}$-differentiable functions, as defined by (72), is again too narrow to be used, say, in field dynamics.

The situation changes radically when we come to consider noncommutative algebras with zero divisors, in particular the algebra of biquaternions $\mathbb{B}$ (quaternions over $\mathbb{C}$). For simplicity let us below restrict ourselves by consideration of the full $N \times N$ matrix algebras over $\mathbb{R}$ or $\mathbb{C}$ (for $N = 2$ we have just an isomorphism $Mat(2, \mathbb{C}) \cong \mathbb{B}$). Then for a *determinant* of the matrix of differentials dF in the l.h.s. of (72) we obtain

$$\det \|dF\| = \det \|L(Z) * R(Z)\| \det \|dZ\| \equiv \lambda(Z) \det \|dZ\|. \tag{73}$$

In the case both L, R are invertible, so that $\lambda(Z) \neq 0$, relation (73) defines a conformal mapping with a scale factor $\lambda(Z)$ and a (positively indefinite or complex) infinitesimal metric *interval* represented by corresponding determinants in (73). Particularly, for $\mathbb{B}$ we deal with conformal mappings in complexified Minkowski space $\mathbb{C}M$.

In a remarkable way, however, for $\det L = 0$ or similarly $\det R = 0$, we have $\lambda(Z) = 0$ and relation (73) defines a reduction of the full vector space of $\mathbb{A}$ to the subspace of null elements (to the complex "light cone" for $\mathbb{B}$). Such mappings may be called *degenerate conformal mappings*; they constitute a wide and important class: in the context of the presented theory just these mappings (differentiable $\mathbb{A}$-functions) are identified with the physical fields.

In the $N \times N$ matrix notation (72) takes the form $(A, B, ... = 1, ...N)$

$$\nabla_{AB} F_{CD} = L_{CA} R_{BD} \tag{74}$$

where ∇_{AB} stands for a derivative operator with respect to the coordinate Z^{AB}. For some indices C, D being fixed we denote $F_{CD} \equiv \Sigma$, $L_{CA} \equiv \phi_A$, $R_{BD} \equiv \psi_B$, and the relation (74) becomes

$$\nabla_{AB}\ \Sigma = \phi_A \psi_B \tag{75}$$

In view of the zero determinant of the matrix in the r.h.s., we get the equation

$$\det \|\nabla_{AB} \Sigma\| = 0, \tag{76}$$

which have to be satisfied for each matrix component $F_{CD} \equiv \Sigma \in \mathbb{R}, \mathbb{C}$ of an $\mathbb{A}$-differentiable function. Equation (76) is a *nonlinear* analogue of the Laplace equation in the complex analysis, and nonlinearity arises here as a direct consequence of the account of noncommutativity in the definition of a differentiable function (72). For the case of biquaternions $\mathbb{B}$ (76) represents a (complexified) 4-*eikonal* equation

$$(\nabla_{00}\Sigma)(\nabla_{11}\Sigma) - (\nabla_{01}\Sigma)(\nabla_{10}\Sigma) = 0 \tag{77}$$

which in Cartesian complex coordinates $z^0, z^3 = z^{00} \pm z^{11}$, $z^1, z^2 = z^{01} \pm iz^{10}$ takes a familiar form

$$(\frac{\partial \Sigma}{\partial z^0})^2 - (\frac{\partial \Sigma}{\partial z^1})^2 - (\frac{\partial \Sigma}{\partial z^2})^2 - (\frac{\partial \Sigma}{\partial z^3})^2 = 0. \tag{78}$$

In the paper we restrict our consideration by a particular, yet a basic subclass of $\mathbb{A}$-differentiable functions for which one of semi-derivatives, say $R(Z)$, is proportional to the function $F(Z)$ itself. Redefining then $L(Z) \to \Phi(Z)$ we get instead of (72)

$$dF = \Phi(Z) * dZ * F(Z). \tag{79}$$

Let now $\{\xi^{(C)}\},\ C = 1, ...N$ be N columns of the matrix $F(Z)$; then we can present (79) in a form of a system of N matrix equations

$$d\xi^{(C)} = \Phi dZ \xi^{(C)} \tag{80}$$

(here and below we omit the symbol of matrix multiplication), which all have to be satisfied with the same (left semi-derivative) matrix $\Phi(Z)$. The quantities $\xi(Z)$ are evidently $SL(N, \mathbb{C})$-spinors with respect to the symmetry transformations of (80) (for $N = 2$ we have demonstrated this in section 2).

Different spinors $\{\xi^{(C)}\}$ may be either functionally dependent or not [12]: in any case an arbitrary solution of (79) may be constructed from (and decomposed into) a set of N solutions $\{\Phi(Z), \xi^{(C)}(Z)\}$ of the system

$$d\xi = \Phi dz \xi. \tag{81}$$

Conversely, from a solution to (81) we easily obtain at least one class of the solutions to the original system (79) by setting N spinors $\xi^{(C)}(Z)$ to be globally proportional to each other (or zero, exept for one of them). Eliminating the semi-derivative matrix Φ from the overdetermined system (81) we come to a nonlinear system for the components of ξ (i.e. of $\mathbb{B}$-differentiable function) only which resembles in a certain sense the Cauchy-Riemann conditions in the complex calculus (for $N = 2$ this is just the system (11) of the paper).

Thus, we have shown that (in a particular case $R(Z) = F(Z)$) the $\mathbb{A}$- differentiable functions, as defined by (72), can all be found from the condition (81). For the algebra of biquaternions $\mathbb{B}$ this equation becomes equivalent to the GSE (1) studied in the paper if we only assume the coordinates to be *real-valued*, so that the matrix $Z \to X,\ X = X^+$ is considered to be hermitian.

This is the only *ad hoc* assumption which is motivated by physical considerations and which does not follow from the algebraic structure or the differentiability conditions (72) themselves (precisely, we have to deal with the full structure of 4-dimensional complex space). Under the assumption made, the coordinate space reduces to the Minkowski one and the whole theory, including the fundamental 4-eikonal equation (78), becomes Lorentz invariant.

Some details and generalizations of the approach afore-presented can be found in the monograph [1] and, partly in English, in [25, 2, 24].

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