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Faithful Representations of Groups by Automorphisms of Topologies

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Abstract. It is proved that every (in general, infinite) group is the full automorphism group of some topology.

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1. INTRODUCTION

In the paper, we generalize the well-known theorem claiming that every finite group is the full automorphism group of some finite graph [1–6]. Namely, we claim that every (in general, infinite) group is the full automorphism group of some topology. In the system of definitions we use, the automorphisms and the isomorphisms of a topology coincide with homeomorphisms, i.e., with topological mappings (we sometimes treat homeomorphisms in a more general way).

Moreover, we consider a special generalization of the notion of topology obtained by weakening the base system of axioms and producing the so-called inductor spaces. The class of these spaces includes not only the ordinary topological spaces but also finite and infinite directed graphs, and also special spaces that are neither topological spaces nor graphs. The application of this notion enables one to immediately generalize the above theorem on automorphisms of graphs to infinite groups and graphs. We also describe the standard actions of some geometric symmetry groups as full automorphism groups of inductor spaces constructed from the corresponding geometric spaces.

In particular, we construct an inductor space on the set of points of a finite-dimensional linear space in such a way that the symmetry group of the inductor space coincides with the Lorentz group provided that the dimension exceeds two. Here the topology induced on every hyperplane cutting out isotropic cones is homeomorphic to the Euclidean one. This result admits a purely geometric formulation. If the dimension of the space is greater than two, then the full automorphism group of the system of cones obtained by parallel shifts of the same spherical cone coincides with the standard action of the Lorentz group (on the space of the corresponding dimension) extended by arbitrary translations, uniform dilations, and also rotations and reflections (Euclidean isometries) in the hyperplane of the spherical section of the cones. The proof uses an earlier result of the author [8] claiming that the affine automorphism group of this system of cones coincides with the group of the above action. We also claim that, if the dimension exceeds two, the automorphisms of the structure are affine.

Representations in the form of automorphisms of an inductor space can also be constructed for the main Euclidean isometries, namely, translations, rotations, and rotations with reflections [11]. The corresponding results are not included in the present paper.

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2. INDUCTOR SPACES

To solve the problem to represent groups by automorphisms of topologies or graphs, it is convenient to introduce an object generalizing these notions and defining a structure on an abstract point set such that the automorphisms of the structure form a representation of the given group. To this end, we introduce the class of inductor spaces.

Definition 2.1. By an induction relation I on a set T we mean an arbitrary set of pairs of the form $[t, V]_I$, where $t \in T$, $V \subset T$. We refer to elements of the induction relation $[t, V]_I$ as *inductor pairs*, the point t is called the *center of induction* or the *center* of the pair, and the set V is called the *inductor* of the pair or of the point t.

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Definition 2.2. By an *inductor space* we mean a set T with an induction relation I satisfying the following axioms.

AI1. *Axiom of membership.* If $[t, V]_I \in I$, then $t \in V$.

AI2. *Axiom of transitivity.* If $[t, V]_I \in I,\ [x, W]_I \in I,\ x \in V$, then $[t, V \cup W]_I \in I$.

In [7–9], inductor spaces were introduced with an extended family of axioms. However, investigations show that the other axioms reduce the class of objects too much and lead to technical complications in the proofs of some theorems. However, the theorems proved below remain valid in the extended axiomatics. The corresponding proofs are presented in [9].

From the point of view of topology, it is natural to interpret an inductor as a neighborhood of the center of induction of a given induction pair. Correspondingly, one can speak about the convergence of sequences and on limit points of subsets on T if an induction relation is given. The axiom of transitivity is in this interpretation a weakened axiom of arbitrary union of open sets. From the point of view of graph theory, one can interpret an inductor as a subset of vertices from which there is a path to the center of induction along the arrows of the graph. Correspondingly, the axiom of transitivity can be interpreted as the possibility to augment the corresponding paths on the graph.

We sometimes use the abbreviated terms "I-relation," "I-pair," and "I-space." On an inductor space, we refer to the induction relation as the *induction*, the set T is called the *support of induction*, and its elements are referred to as *points of the space*. This term arose in the use of I-spaces as supports of distributed processes in mathematical models, where inductors of a point play the role of domains of influence on the point [8, 9]. This interpretation is inessential for the purposes of the present paper.

In the general case, inductor spaces are neither graphs nor topological spaces. For the corresponding examples, see [8, 9]. However, topological spaces and graphs are simplest special cases of I-spaces. For exact definitions, see [8, 9] and the definitions given below.

3. RELATIONS OF INDUCTOR SPACES TO GRAPHS AND TOPOLOGICAL SPACES

A topology τ given by a family of open subsets on a set of points T defines an inductor space formed of all possible I-pairs of the form $[x, V]_\tau$, where $V \in \tau,\ x \in V$. In this case, to any topological neighborhood of an arbitrary point there corresponds an I-pair with the center of induction at a given point. To any open set, the set of I-pairs corresponding to diverse centers is assigned.

Here it should be taken into account that, when constructing the topology from a base family of neighborhoods of the points, one uses a broader class of generating operations (arbitrary unions and finite intersections) than that for the inductor spaces (with transitive unions only). Therefore, the base family of I-pairs generating the topology in the form of an inductor space corresponds in the general case to an extended family of base neighborhoods.

An important difference between the general construction of an induction from a topology is that a point can belong to an inductor which is not a proper neighborhood of the point, i.e., there is no I-pair for which the given point is the center of induction in the inductor. However, when passing from topology to an induction in the above way, this situation cannot occur.

A directed graph with a set of vertices T (defined as a given subset $S \subset T \times T$ of the ordered pairs of vertices ("arrows") $(x, y) =$ (initial vertex, ending vertex)) can be described as an inductor space on the set of vertices as points (the base I-pairs are of the form $[x, V]_S$, where x is a vertex and V is the set of vertices issuing arrows ending at x, $V = \{y | (y, x) \in S\}$). The other elements of induction are defined by the base elements by the axiom of transitivity.

Definition 3.1. The above constructions of inductions from topologies and graphs are said to be *canonical*.

Corollary 3.1. *For a graph, a vertex y enters some inductor of a vertex x if and only if there is a path from y to x along the arrows.*

Corollary 3.2. *Both for topologies and for graphs, under the canonical imprinting* (*passage to the corresponding inductions*), *the intersection of any two inductors of a point gives an inductor of the same point. A more general property holds*: *if some inductor of a point x belongs to an inductor V of some point y, then any inductor of x intersects V by an inductor of x.*

Other versions of correspondence are possible. Their common property is the possibility to uniquely recover the topology or a graph from the induction, and conversely. A version of passage from a topology to an I-space by using closures of open sets as inductors was suggested in [7–9]. This was convenient to describe models of distributed processes. In this version, limit points for sets and sequences must be defined by using interiors of inductors as neighborhoods of points. We do not consider this version in the present paper.

Definition 3.2. By an *alternative passage* from a graph to an induction we mean base I-pairs $[x,\{x;y\}]_{Sa},\ (x,y)\in S$. These must also be completed by I-pairs obtained by the closure according to the axiom of transitivity.

In general, this version gives more inductors of a point than the canonical one. Corollary 3.2 can fail for the alternative passage, but Corollary 3.1 is preserved.

Some topological notions can be extended to general inductor spaces.

Definition 3.3. A point is said to be *interior* for some subset of the support of induction if this subset contains some inductor of the point. The other points of this subset are said to be *boundary*. A point is *limit for a subset* of the support if any inductor of the point contains an element of the subset. A point is *limit for a sequence* of points in the support of induction if any inductor of the point contains infinitely many terms of the sequence. A point is a *limit of a sequence* if any its inductor contains all terms of the sequence starting from some of them.

4. AUTOMORPHISMS OF INDUCTOR SPACES

The class of inductor spaces can be equipped with the notion of isomorphism.

Definition 4.1. Two I-spaces $[T,I],\ [T',I']$ are said to be *isomorphic* if there is a bijection of the supports (an *isomorphism*) $h\colon T\leftrightarrow T'$ generating a bijection of the inductions $H\colon I\leftrightarrow I'$, where $H\circ[t,V]_I\overset{\text{def}}{=}[h(t),h\circ V]_{I'}$. An isomorphism of an I-space onto itself ($T=T',\ I=I'$) is called an *automorphism*.

Definition 4.2. The automorphisms of any I-space $[T,I]$ form a group with respect to the operation of superposition. Denote it by $\operatorname{aut}[T,I]$ (the *automorphism group*). The unit element is the identity automorphism $E(x)=x,\ x\in T$. The element inverse to an arbitrary automorphism h is the bijection h^{-1}.

Lemma 4.1. *Homeomorphisms of topologies and isomorphisms of graphs are equivalent to isomorphisms of inductor spaces under the canonical mapping.*

Proof. If there is a homeomorphism of two topologies, then the image and the preimage of any open set in one of the topologies is open in the other. Under the canonical mapping into an inductor space, the inductors are bijective images of open sets and the corresponding centers of induction are bijective images of points of these sets. Thus, every homeomorphism of two topologies is an isomorphism of their canonical imprints. Conversely, an isomorphism of two canonical imprints of topologies is a bijection of the sets of their supports under which every inductor of one of the inductions is the image (and the preimage) of an inductor of the other induction. Hence, every open set of one of the topologies is the image and the preimage of an open set of the other, and hence the mapping is a homeomorphism. Thus, we are done for topologies.

An isomorphism of two graphs is a bijection of the sets of their vertices under which the arrows of one of the graphs are unique images and preimages of the arrows of the other graph. This means that the minimal neighborhood of any vertex with respect to the incoming arrows of one of the graphs corresponds to the minimal neighborhood of the image of this vertex with respect to the incoming arrows of the other graph. Since these neighborhoods form a generating system of I-pairs of each of the inductions, it follows that an isomorphism of graphs is an isomorphism of their canonical imprints. Conversely, if there is an isomorphism of canonical inductor imprints of two graphs, then it defines a bijection of the supports of these I-spaces, and the supports coincide with the sets of vertices. Every isomorphism of inductor spaces preserves an embedding of inductors of any point because it is a bijection. Hence, if a point has an inductor that is minimal with respect to inclusion, then it is mapped onto the minimal inductor of the image of the point (both under the direct and the inverse mapping). Hence, an isomorphism of canonical imprints of two graphs assigns to incoming arrows (of any vertex) of one of the graphs incoming arrows (of the image of this vertex) on the other graph. This means that the image and the preimage of any arrow is an arrow of the corresponding orientation, which defines an isomorphism of graphs because the mapping is bijective.

Remark 4.1. Lemma 4.1 remains valid for the alternative passage from a graph to an I-space because the images of the arrows in this graph are minimal inductors of their issuing vertices. In this case, a point can have arbitrarily many minimal inductors. For a canonical mapping (of a graph of a topology), at most one minimal inductor can occur.

5. RIGID I-SPACES, TOPOLOGIES, AND GRAPHS

Definition 5.1. An inductor space is said to be *rigid* if its automorphism group consists of the identity automorphism only, $\mathrm{aut}[T, I] = \{E\}$.

The rigid spaces form an infinite class. This enables one to extend inductor spaces in such a way that their automorphism groups are preserved or even strictly reduced. Below we suggest a standard construction of infinitely many pairwise nonisomorphic rigid spaces which is used in the proof of the main theorems.

Consider an arbitrary transfinite order type p. To this type, there corresponds a well-ordered set of the related cardinality $\#p$, which is a transfinite sequence of order type p of pairwise distinct elements,

$$S(p) = (a_1,\, a_2, \ldots , a_p)\ . \tag{5.1}$$

If p is a limit ordinal, then the last term in (5.1) must be understood conditionally, as a restriction of the sequence. Assign to the sequence (5.1) the following inductor space $SH_p = [S(p), H(p)]$ with the support $S(p)$ and the induction $H(p)$,

$$H(p) = \{[a_i, \{a_1;\, a_2; \ldots ; a_i\}]_H \mid i \leqslant p\}\ . \tag{5.2}$$

In the induction (5.2), the only inductor of any point i is the set $S(i) \subset S(p)$ of the elements in (5.1) that do not exceed i in this ordering. The spaces $SH(p)$ are not isomorphic for distinct ordinals p because there is no order-preserving bijection for transfinite sequences of distinct order types. For the same reason, the automorphism group of $SH(p)$ is trivial. Every initial segment $S(i)$ in $S(p)$ (an inductor) can be strictly monotonically mapped only onto itself, and thus $\mathrm{aut}(SH(p)) = \mathrm{aut}[S(p), H(p)] = \{E\}$.

Lemma 5.1. *For any cardinality m, there is a rigid inductor space whose support is of cardinality m. To any ordinal p of cardinality m, there corresponds a rigid I-space nonisomorphic to any space of this kind for other ordinals.* (The proof was given above.)

Remark 5.1. In what follows, we need a set of G copies of some rigid space that are isomorphic but distinguishable. To this end, we introduce a parameter $g \in G$. Copies of the space are denoted by $SH_{p,g} = [S(p, g), H(p, g)]$ and their points by $a(g)_i$, in accordance with (5.1).

One can pass from rigid I-spaces to rigid topologies and graphs.

Definition 5.2. A graph is said to be *rigid* if its automorphism group contains the identity automorphism only.

Lemma 5.2. *For any cardinality m, there is a rigid graph with support of cardinality m. To any ordinal p of cardinality m, there corresponds a rigid graph nonisomorphic to these graphs for other ordinals.*

Proof. The above I-space SH_p is the canonical imprint of the graph SHG_p with the set of vertices $S(p)$ and the arrows of the form (a_i, a_j), where $1 \leqslant i \leqslant j \leqslant p$. The assertion of the lemma follows from Lemma 4.1 and Lemma 5.1.

Definition 5.3. A topological space is said to be *rigid* if the group of its automorphisms (self-homeomorphisms) contains the identity automorphism only.

Lemma 5.3. *For any cardinality m, there is a rigid topological space with support of cardinality m. To any ordinal p of cardinality m, there corresponds a rigid topological space not homeomorphic to the topological spaces corresponding to the other ordinals.*

Proof. The inductors of the above I-space SH_p give a base system of neighborhoods for the canonical imprint of the topological space SHT_p constructed on the point set $S(p)$ by using the open sets of the form $S(j)$, $1 \leqslant j \leqslant p$. The set $S(i)$ is the minimal neighborhood of the point a_i and the minimal inductor of the canonical imprint of a_i. Under an automorphism of inductions, the minimal inductor of a point passes to the minimal inductor of its image. Therefore, every automorphism of SHT_p is an automorphism of SH_p. Thus, the assertion of the lemma follows from Lemma 4.1 and Lemma 5.1.

6. REPRESENTATIONS OF GROUPS BY AUTOMORPHISMS OF I-SPACES, TOPOLOGICAL SPACES, AND GRAPHS

As was proved in [8, 9], every group defines an infinite class of inductor spaces whose automorphism groups are (algebraically) isomorphic to the given group. We refer to these spaces as imprints of the group (giving the class of imprints), and their automorphisms can be regarded as representations of the group. The induced action of automorphisms on the linear space of functions defined on the support of an I-space gives a ("multiplicative") representation into an algebra of linear operators. To obtain a similar result for automorphism groups of topological spaces and graphs, below we use the construction of inductor imprints of groups.

Definition 6.1. An inductor space is said to be an *inductor imprint* (an *I-imprint*, an *imprint*) of some group if the full automorphism group of the space is isomorphic to the given group.

Theorem 6.1. *Every group G admits an inductor imprint.*

Proof. Let us use the notation S_p, H_p, and SH_p introduced in the proof of Lemma 5.3. Denote by m the cardinality of the set of elements of the group G. Let $Q = \{Y_g | g \in G \cup \{0\}\}$ be a family of disjoint sets (layers), the cardinality of each of the layers Y_g being equal to m. Choose an ordinal p of cardinality m. Define a bijection $r\colon G \leftrightarrow S_p$. On any set $Y_g \in Q$, we introduce a bijection $h_g\colon Y_g \leftrightarrow G$. In this case, a bijection $r \circ h_g = r_g\colon Y_g \leftrightarrow S_p$ is well defined. Let us equip every layer Y_g, $g \in G$, with the induction $I_g = H_p$ with respect to the order r_g. Define the induction $I_0 = \{[x, \{x\}]_I | x \in Y_0\}$ on the set Y_0; we refer to I_0 as a *loop induction* because I_0 canonically corresponds to a graph in which every vertex gives an arrow to itself and there are no other arrows. Write $T = \cup Q$. The induction I on T, along with the above inductors in $\cup I_g|_{g\in G} \cup \{0\}$, contains all inductors of the form $[x, \{x; z\}]_I$, $x \in Y_0$, $z = h_g^{-1}(h_0(x)g^{-1})$, $g \in G$. This corresponds to arrows of the graph J joining the point x of the layer Y_0 (as the entrance point) to the point z of the layer Y_g (as the issuing point), and z corresponds on the layer Y_g to the right multiplication by $g^{-1} \in G$ of the image of the point x in the system of mappings h of these layers onto the group G.

Let us show that $\mathrm{aut}[T, I] \simeq G$. The induction on every layer Y_g, $g \in G$, is rigid. Hence, under any automorphism, this layer can be mapped only onto a layer of the same form by preserving the order of r. The layer Y_0 can be mapped only onto itself, and admits arbitrary self-bijections with respect to the induction I_0. However, these self-bijections are limited by the graph J. The only admissible auto-bijections must take the layers Y_g, $g \in G$, bijectively onto one another by preserving the arrows in J. Let us show that every bijection of this kind corresponds to the right multiplication of the indices of the layers by some element of the group. Let $v \in \mathrm{aut}[T, I]$, $x \in Y_0$. For any x, there is an $f \in G$, $f = (h_0(x))^{-1} h_0(v(x))$, for which $v(x) = h_0^{-1}(h_0(x)f)$. Consider an arbitrary layer Y_g, $g \in G$. Suppose that $v \circ Y_g = Y_q$. There is a unique arrow (x, z_g), $z_g \in Y_g$, in the graph J, namely, $z_g = h_g^{-1}(h_0(x)g^{-1})$. Since v is an automorphism, it follows that J contains an arrow $(v(x), v(z_g))$, where $v(z_g) = h_q^{-1}(h_0(v(x))q^{-1}) = h_q^{-1}(h_0(x)fq^{-1})$. Since $v\colon Y_g \leftrightarrow Y_q$ is an isomorphism, it follows that $r(h_g(z_g)) = r(h_0(x)g^{-1}) = r(h_q(v(z_g))) = r(h_0(x)fq^{-1})$. Since r is bijective, we have $h_0(x)g^{-1} = h_0(x)fq^{-1}$ and $q = gf$.

Since the entire layer Y_g is taken onto the layer Y_q, it follows that the element f of the group is the same for any $x \in Y_0$, namely, $f = g^{-1}q$. The automorphism v on the layer Y_0 is of the form $v \circ Y_0 = h_0^{-1} \circ ((h_0 \circ Y_0)f)$. If $v_1, v_2 \in \mathrm{aut}[T, I]$, and if the elements $f_1, f_2 \in G$ correspond to v_1 and v_2 by the above formula, then $v_2 \circ v_1 \circ Y_0 = h_0^{-1} \circ ((h_0 \circ Y_0)f_1 f_2)$. This formula establishes an isomorphism $\mathrm{aut}[T, I] \simeq G$. Thus, the automorphisms define a representation of the group.

Remark 6.1. If one identifies the elements of the layers Y_g with the elements of the group G with respect to the bijections h_g and if e is the identity element of G, then, on the layers with the indices $g \in G$, the graph J writes out a permutation of the elements of the layer Y_e which arises under the right action of the element g^{-1} on the group. Therefore, an automorphism is admissible only if a self-action of the group corresponds to the automorphism. This excludes the outer automorphisms of the group from $\mathrm{aut}[T, I]$. In fact, the above imprint of the group in the finite case corresponds to the representation of all permutations of the elements of the group arising under a multiplication by an element of the group in the form of a graph.

Claim 6.1. *The nonisomorphic imprints of a given group form an infinite class.*

Proof. The assertion follows from Lemma 5.1. Consider the union of an I-space T and a rigid space X disjoint from T and such that X has no isomorphic embeddings in T. If now new inductors are added, then $\mathrm{aut}(T) \simeq \mathrm{aut}(T \cup X)$. At the same time, adding nonisomorphic spaces X gives nonisomorphic unions.

Theorem 6.2. *Every group has an imprint in the class of graphs* (*in general, infinite*).

Proof. Consider the construction in the proof of Theorem 6.1. By Lemma 5.2, the rigid layers Y_g can be constructed in the form of graphs. Their union with the graph Y_0 by using the graph J gives a desired graph with the automorphism group $\mathrm{aut}[T, I] \simeq G$.

Remark 6.2. If a group is finite, then the imprint constructed above is a finite graph.

Thus, the above construction gives another proof of the theorem concerning representations of finite groups [2].

Theorem 6.3. *Every group G has an imprint in the class of topological spaces.*

Proof. Consider the construction in the proof of Theorem 6.1. By Lemma 5.3, the rigid layers Y_g can be constructed in the form of topological spaces. The layer Y_0 can then be equipped with the discrete topology. The arrows of the graph J are transformed into open sets not canonically. This is needed to prevent the occurrence of new neighborhoods on the layers Y_g. Denote by $W_g(x)$ the set of elements on Y_g which do not exceed x in the ordering of S_p. An arrow (x, y), where $x \in Y_g$ and $y \in Y_0$, is transformed to the open set $V_{x,y} = \{y\} \cup W_g(x)$. In this case, the layers remain rigid because they keep the intrinsic topology of Lemma 5.3. New neighborhoods of points occur on the layer Y_0 due to unions of open sets $V_{x,y}$. However, the correspondence of the points of the layers is preserved as in the graph J, because $V_{x,y}$ is the minimal neighborhood of the point y having an intersection with the layer containing x. Therefore, under any automorphism, $V_{x,y}$ comes to a neighborhood of the same form. These neighborhoods of the point $y \in Y_0$ (on every layer) uniquely define the arrow (x, y). We obtain a topological space with the automorphism group $\mathrm{aut}[T, I] \simeq G$.

Remark 6.3. If a group is finite, then the imprint thus obtained is a finite topological space.

7. REPRESENTATIONS OF AN ACTION OF A GROUP BY AUTOMORPHISMS OF I-SPACES, TOPOLOGICAL SPACES, AND GRAPHS

Definition 7.1. By an *action* $W : G$ of some group G on a given set W we mean a family Q of self-bijections of W, $q \in Q \Rightarrow q : W \leftrightarrow W$, which is closed under superposition and group isomorphic (with respect to the superposition) to the group G, i.e., there is a bijection $P \colon Q \leftrightarrow G$ such that $q, q' \in Q \Rightarrow P(q(q')) = P(q')P(q)$.

Definition 7.2. An inductor space $[T, I]$ is said to be an *inductor imprint* (an *I-imprint*, an *imprint*) of an action of a group G on a set W if there are an injection $H \colon W \to\to T$ such that HW is an invariant set of all automorphisms of the space T and a bijection $P \colon (\mathrm{aut}[T, I]/_{HW}) \leftrightarrow G$ for which $(H^{-1} \times P) \circ (HW \times \mathrm{aut}[T, I]/_{HW}) = (W : G)$. In this case, we write $\mathrm{aut}[T, I]/_{HW} \simeq (W : G)$. The set in the subscript under the slash stands for the restriction of the action of the automorphisms to this set.

Theorem 7.1. *Every action of a group G on a set W has an inductor imprint.*

Proof. The construction of the imprint is similar to the construction in Theorem 6.1. The layers Y_g correspond to elements g of the group G. The cardinality of the ordinal p and of every layer Y_g, $g \in G \cup \{0\}$, is equal to $m = \#W$. Introduce some bijections $r \colon W \leftrightarrow S_p$ and $h_g \colon Y_g \leftrightarrow W$. The layers Y_g, $g \in G$, have rigid induction H_p in the ordering h_g. The layer Y_0 is equipped with the loop induction. Denote by $x \colon g = y$, $x, y \in W$, $g \in G$, the action of an element of a group on an element of the set. The graph J is constructed on $T = \cup Y_g | g \in G$ by the arrows (z, x), where $x \in Y_0$ and $z = z(g, x) = h_g^{-1}(h_0 x : g^{-1}) \in Y_g$. To an arrow (z, x), the inductor pair $[x, \{x; z\}]_I$ corresponds. This completes the construction of the induction I. If $v \in \mathrm{aut}[T, I]$ and $x \in Y_0$, then the value $v(x)$ must satisfy the condition $h_g(z(g, x)) = h_q(z(q, v(x)))$ for some layers Y_g and Y_q. In this case, there is an $f = g^{-1}q$ for which $v(x) = x : f$. Here the entire layer Y_g is bijectively mapped onto the layer Y_q because the induction H_p is rigid. Therefore, for any $x \in Y_0$, the action of $v \in \mathrm{aut}[T, I]$ is defined by the formula $v(x) = x : f$. If $v_1, v_2 \in \operatorname{aut}[T, I]$ and if $f_1, f_2 \in G$ correspond to v_1 and v_2 by the above formula, then $v_2 \circ v_1 \circ Y_0 = h_0^{-1} \circ (((h_0 \circ Y_0) : f_1) : f_2) = h_0^{-1} \circ ((h_0 \circ Y_0) : (f_1 f_2))$. This formula establishes an isomorphism $\operatorname{aut}[T, I] \simeq (W : G)$. The image of the set W is the set Y_0 with the injection h_0^{-1} and a bijection of the group $v \mapsto f$.

Theorem 7.2. *Any action of the group G on a set has an imprint in the class of topological spaces and in the class of graphs.*

The proof is similar to the proofs of Theorems 6.3 and 6.2 and uses the construction in the proof of Theorem 7.1.

Definition 7.3. By an *action* $W : G\,(\mathrm{inv} = U)$ of some group G on a given set W with an *invariant system of subsets* U $(V \in U \Rightarrow V \subset W)$ we mean a family Q of self-bijections such that $q \in Q \,\&\, V \in U \Rightarrow q : W \leftrightarrow W \,\&\, q \circ V \in U$ and Q is closed with respect to superposition and a group isomorphic to G with respect to superposition, i.e., there is a bijection $P \colon Q \leftrightarrow G$ for which $q, q' \in Q \Rightarrow P(q(q')) = P(q')P(q)$.

In other words, an action of a group (on a set) with an invariant system of subsets is an action of the group on the set such that every self-bijection takes any subset in a system into a subset in the same system.

Definition 7.4. By the *induced induction* I/A on a subset A of an inductor space $[T, I]$ we mean the set of inductor pairs $\{[x, B]_{I/A} \,|\, x \in A,\ B = C \cap A,\ [x, C]_I \in I\}$.

Definition 7.5. An inductor space $[T, I]$ is said to be an *inductor imprint* (an *I-imprint*, an *imprint*) of an action of a group G on a set W with invariant system of subsets U if it is an imprint of the action with an injection H and the images of the subsets in U are a generating system of inductors for the induced induction on HW. In this case, we write $\operatorname{aut}[T, I]/_{HW} \simeq W : G(\mathrm{inv} = U)$.

Lemma 7.1. *An action of a group on a set with an invariant system of subsets* $W : G(\mathrm{inv} = U)$ *is an action* $W : G(\mathrm{inv} = \mathrm{tr}(U))$, *where* $\mathrm{tr}(U) = \{A \cup B \mid A, B \in U,\ A \cap B \neq \varnothing\}$.

Proof. Any bijection takes a union of sets into the union of their images and any intersection of sets into the intersection of their images. Therefore, $\mathrm{tr}(U)$ is invariant together with U with respect to any system of bijections.

Lemma 7.2. *Any action of a group G on a set W with an invariant system of subsets U admits an inductor imprint.*

Proof. Let us use the construction in the proof of Theorem 7.1. The image of HW in this I-imprint coincides with the layer Y_0. This layer is equipped with the loop induction. Therefore, this is not a desired imprint in general. Let us add another layer with a nongroup index Y_W and with the corresponding bijection $h_W \colon Y_W \leftrightarrow W$ for $T = \cup Y_c \,|c \in G \cup \{0; W\}$; $T' = \cup Y_c \,|c \in G \cup \{0\}$. Let us complete the graph J with the arrows $(w, x),\ w \in Y_W,\ x \in Y_0,\ h_W(w) = h_0(x)$, where

$$w = w(x) = h_W^{-1}(h_0(x)). \tag{7.1}$$

Here the induced loop induction on the layer Y_0 is preserved. Write $H = h_W^{-1}$. On the layer Y_W, define the I-pairs of the form $[w, V]_I$, where $V \in H \circ U$ and $w \in V$. Let us complete them by the inductors of closure by the axiom of transitive union. By construction, the family of inductors of this induction coincides with $\mathrm{tr}(U)$. By Lemma 7.1, this system of inductors is also invariant with respect to the action of the group. In this case, the self-bijections of the layer Y_W correspond under the automorphisms of $[T, I]$ (by the mapping (7.1)) to the bijections of the layer Y_0 in the automorphisms of $[T', I]$ given in Theorem 7.1. No automorphism can take a point of the layer Y_0 to a point of the layer Y_W and conversely, because the inductors of the layer Y_W are disjoint from the layers Y_g, $g \in G$. If the group G is nontrivial, then the layer Y_W is not isomorphic to the layers Y_g, $g \in G$, and they cannot be mapped into one another under automorphisms. Therefore, the automorphism group of the space this constructed is isomorphic to the automorphism group of the space in Theorem 7.1, and hence isomorphic to G. Using the mapping H as an injection $H \colon W \rightarrow\!\!\rightarrow T$, we obtain the desired imprint. Finally, if a homeomorphism of Y_W and one of the layers Y_g, $g \in G$, exists, then the group G is trivial, $g = e$, because these layers are rigid. In this case, one should delete the layer Y_W from the construction and set $H = h_e^{-1}$. This trivially solves the problem of imprint for the identity action.

Theorem 7.3. *Every subgroup G of the homeomorphism group of a topological space $(X|\tau)$ admits an inductor imprint with an invariant system of open sets. These imprints can be constructed in the class of topological spaces.*

Proof. The existence of an imprint follows from Lemma 7.2, $W = X$; $U = \tau$; $H = h_W^{-1}$; $H' = h_0^{-1}$. One cannot pass to an imprint in the class of graphs in general by using an argument similar to that in Theorem 6.2 because the induction on the layer Y_W in this construction is a topological space and its reduction to a graph is impossible in general; however, the passage to the class of topological spaces is similar to Theorem 6.3. To this end, one must extend the family of open sets obtained in Theorem 6.3 by the family of sets of the form $U_H = \{u_V = H \circ V \cup H' \circ V \mid V \in \tau\}$, where $H \circ V \subset Y_W,\ H' \circ V \subset Y_0$, together with the sets generated by the above sets by the axioms of arbitrary union and finite intersection. Since a topology is closed with respect to these operations, it follows that there are no new inductors on the layer Y_W under the canonical mapping of the topology. On the layer Y_0, under the passage from the loop induction to the discrete topology, according to Theorem 6.3, all inductors of the form $[x, v]_I,\ v \subset Y_0$, where $x \in v$, necessarily occur, but these do not influence the automorphism group because they are invariant with respect to any bijection. The new induced inductors of the form $H' \circ V$ belong to this class of subsets and do not change the induced induction $[T, I]/_{Y_0}$. Therefore, the set of inductors $[T, I]/_{Y_0 \cup Y_W}$ thus obtained is invariant with respect to the action of automorphisms representing the group G. The induced induction is preserved on the other layers. Therefore, if the layer Y_W is not homeomorphic to any layer of the form Y_g, $g \in G$, then the property of inductor imprint of the action $X : G(\operatorname{inv} \tau)$ is preserved. The case in which the layer Y_W is homeomorphic to one of the layers Y_g, $g \in G$, is treated as in the proof of Lemma 7.2.

8. SPACES WITH CONIC INDUCTION

Definition 8.1. By a *conic space* $\mathrm{Rc}[n]$ we mean a space $\mathbb{R}^n$ with an induction for which the generating system of inductors of a point y are cones representable in a system of Cartesian coordinates $x_1, \dots, x_n$ in the form $0 \leqslant (x_1 - y_1)^2 - (x_2 - y_2)^2 - \dots - (x_n - y_n)^2$, $0 \leqslant y_1 - x_1 \leqslant H$, $H > 0$.

The assumption $|y_1 - x_1| \leqslant H$ leads to the so-called *biconic* space $\mathrm{Rb}[n]$, the condition $0 \leqslant y_1 - x_1$ to the so-called *full conic* space $\mathrm{Rfc}[n]$, and, if no conditions on $y_1 - x_1$ are imposed, then the space is referred to as a *full biconic* space $\mathrm{Rfb}[n]$.

The corresponding automorphism groups are denoted, according to the structure of inductors, by $\operatorname{aut} \mathrm{Rc}[n]$, etc.

A conic space can be regarded as a vector space equipped with a system of spherical cones obtained as all possible parallel shifts of a chosen cone. In biconic spaces, the cones are two-sided with respect to the vertex. In nonfull spaces, along with full cones, cones of bounded height are also considered. The parameter H defines this condition for a specific cone in the generating set. The transitive union can produce diverse bounds of nonfull cones. The conic inductor spaces do not belong to classes of graphs or topological spaces.

Remark 8.1. Using the axiom of transitive union, one can readily show that $\operatorname{aut} \mathrm{Rc}[n] = \operatorname{aut} \mathrm{Rfc}[n]$ and $\operatorname{aut} \mathrm{Rb}[n] = \operatorname{aut} \mathrm{Rfb}[n]$.

Therefore, it suffices to study the automorphism groups for the classes of full spaces only. Distinguishing of the above four classes of spaces of conic type is related to applications to mathematical modeling. The use of bounded cones (with the parameter H) defines a topology on the time axis in models of mathematical physics [8, 12]. However, this is inessential for the purposes of the present paper. In what follows, we consider only full conic and biconic spaces. We are interested in their automorphism groups.

As was proved in [8], the affine automorphism group of a conic space coincides with the canonical action of the Lorentz group (on the Minkowski space of corresponding dimension) extended by all parallel translations, rotations, and reflections in the hyperplane of the spherical section of the cone, and also by uniform dilations (multiplications of vectors by a positive number). Below we refer to this action as an affine extension of the (action of the) Lorentz group and denote it by $\mathrm{ALor}[n]$ and denote the abstract group by itself by $\mathrm{GAL}[n]$.

Below we show that, for the dimensions 1 and 2, the full automorphism group is significantly larger than its affine subgroup, whereas the full automorphism group coincides with the affine subgroup beginning with the dimension 3.

Denote by $\circ$ the superposition of two actions on a set, by $\times$ the direct product of groups, by $\otimes$ the direct product of actions on the direct product of the corresponding sets (in the linear space this is reduced to the direct sum $\oplus$ of the subspaces), and by $\mathrm{Gaut}[T, I]$ an abstract group isomorphic to the automorphism group of the space.

Theorem 8.1. *The group* $\mathrm{aut}\,\mathrm{Rc}[1]$ *is formed by all positively monotone bijections of* $\mathbb{R}^1$, *the group* $\mathrm{aut}\,\mathrm{Rc}[2]$ *is* $((\mathrm{aut}\,\mathrm{Rc}[1] : Z_1) \otimes (\mathrm{aut}\,\mathrm{Rc}[1] : Z_2)) \circ h_2(2)$, *where* Z_1 *and* Z_2 *stand for two generators of the two-dimensional cone, the action* $h_n(i)$, $i \leqslant n$, *on* $\mathbb{R}^n$ *is the inversion* (*the multiplication by* -1) *of the* x_i *axis, and, in particular,* $h_2(2)$ *makes a permutation of the generators of the cone on* $\mathbb{R}^2$, *the group* $\mathrm{Gaut}\,\mathrm{Rc}[2]$ *is* $\mathrm{Gaut}\,\mathrm{Rc}[1] \times \mathrm{Gaut}\,\mathrm{Rc}[1] \times S_2$, *where* $S_2 = Gh_n(i)$ *stands for the group of order two,* $\mathrm{aut}\,\mathrm{Rc}[n] = \mathrm{ALor}[n]$ *for* $n \geqslant 3$, *and* $\mathrm{Gaut}\,\mathrm{Rc}[n] = \mathrm{GAL}[n]$ *for* $n \geqslant 3$.

Theorem 8.2. *The group* $\mathrm{aut}\,\mathrm{Rb}[1]$ *is formed by all monotone bijections of* $\mathbb{R}^1$, $\mathrm{aut}\,\mathrm{Rb}[1] = \mathrm{aut}\,\mathrm{Rc}[1] \circ R_h(1)$, $\mathrm{GRb}[1] = \mathrm{GRc}[1] \times S_2$, *where the action of* $R_h(n)$ *is the reflection of* $\mathbb{R}^n$ *with respect to the origin* (*the multiplication of vectors by* -1), $\mathrm{aut}\,\mathrm{Rb}[2] = \mathrm{aut}\,\mathrm{Rc}[2] \circ h_2(1) \circ h_2(2)$, $\mathrm{Gaut}\,\mathrm{Rb}[2] = \mathrm{Gaut}\,\mathrm{Rc}[1] \times \mathrm{Gaut}\,\mathrm{Rc}[1] \times S_2 \times S_2$, $\mathrm{aut}\,\mathrm{Rb}[n] = \mathrm{ALor}[n] \circ R_h(n)$ *for* $n \geqslant 3$, *and* $\mathrm{Gaut}\,\mathrm{Rb}[n] = \mathrm{GAL}[n] \times S_2$ *for* $n \geqslant 3$.

Proof. The actions of the groups $\mathrm{aut}\,\mathrm{Rb}[n]$ and $\mathrm{aut}\,\mathrm{Rc}[n]$ differ only by the direct additional multiplication by the inversion of the axis of the cone, which corresponds to the coordinate x_1 (this is conditioned by the corresponding symmetry of the generating system of biconic induction). Therefore, Theorem 8.2 immediately follows from Theorem 8.1. The relations for the abstract groups immediately follow from relations for the actions of automorphism groups. For this reason, it suffices to carry out the proof of the relations for the automorphisms of the conic induction.

Case $n = 1$. In this case, the conic space is the "directed line:" a neighborhood of a point is the left half-line in which this point is the right end. The continuous mappings in this induction are left continuous functions. An automorphism which is a continuous self-bijection, can be only an arbitrary strictly monotone continuous real function which is unbounded in all directions.

Case $n = 2$. In this case, the cone (the inductor of a point) is an angle with a vertex at the point. The generators of the cone are the sides of the angle. The corresponding generators of all cones (the inductors) are parallel. Under an automorphism, any generator of any cone passes to a generator of the image of the cone. The mapping can take a generator either to the corresponding generator or to the opposite one. Let us pass to the system of coordinates whose axes are parallel to the generators with the origin at some point $(0, 0)$ and with directions e and t. On the lines of these vectors, the induction is induced by the directed lines. If U is an automorphism, then either $U(x, y) = U(xe + yt) = U(0, 0) + V(x)e + W(y)t$, where V and W are automorphisms of the directed lines or $U(x, y) = U(xe + yt) = U(0, 0) + V(x)t + W(y)e$, where V and W are mutual homeomorphisms of the lines. With regard to the above set of automorphisms of a directed line, this corresponds to the assertion of the theorem.

Case $n = 3$. To prove the assertion of the theorem, it suffices to show that all automorphisms are affine, i.e., that the class of lines is preserved under the action (this is really sufficient because the group of affine automorphisms of $\mathrm{Rc}[n]$ coincides with the extended action of the Lorentz group [8]), or, in other words, that every automorphism takes any line to a line. Introduce some notions.

An I-*cone* is a cone which is the union of all inductors of a single point in $\mathrm{Rc}[3]$ (the inductor of a full conic induction). A C-*line* is a line continuing a generator of an I-cone. A C-*plane* is a plane tangent to some I-cone. A T-*line* is a line partially belonging to the interior of an I-cone. A T-*plane* is a plane formed by lines parallel to a T-line. A P-*plane* is a secant plane for an I-cone. A P-*line* is a line belonging to a P-plane. Correspondingly, we generally speak of C-*objects*, T-*objects*, and P-*objects*.

Lemma 8.1. *Any automorphism of a conic space* $\mathrm{Rc}[n]$ *is continuous in the topology of* $\mathbb{R}^n$.

Proof. Suppose that a sequence of points $r(1), r(2), \dots$ converges to a point r. In this case, one can construct a sequence of embedded I-cones $(K(i)|i = 1, \dots)$ with vertices at some points $g(1)$, $g(2)$, $\dots$, and with heights $H(1)$, $H(2)$, $\dots$, where $\lim_{i\to\infty} g(i) = r$ and $\lim_{i\to\infty} H(i) = 0$ and all points $r(j)$, $j \geqslant i$, belong to the interior of the cone $K(i)$. The only common point of all cones is the point r. Applying an arbitrary automorphism u to the I-cones $K(1)$, $K(2)$, ... , we obtain an embedded system of I-cones $(u \circ K(i) | i = 1, \dots)$ with a single common point $u(r)$. Here the images $u(r(j))$, $j \geqslant i$, are placed inside the I-cone $u \circ K(i)$. This implies that $\lim_{i\to\infty} u(r(i)) = u(r)$.

Lemma 8.2. *Every P-line is the intersection of two C-planes.*

Proof. Consider a P-line L. Choose a point r on L and place the vertex of the I-cone K having no other intersection points with L there. Draw two distinct tangent planes to the cone that contain L (recall that the dimension of the space is not less than three). Since every plane tangent to an I-cone is a C-plane, this will prove the lemma. Let us prove that these planes always exist for a cone and for a line passing through the vertex from outside. Consider the plane V of the circular cross-section of the cone K. Two cases are possible: the line L can be either parallel or not parallel to V. If the line L is parallel to the plane, then it is orthogonal to the axis of the cone. In this case, there are two generators of the cone K, which are orthogonal to the line L. The planes spanned by these generators and the line L are the desired tangent planes. If the line is not parallel to the plane, then they have an intersection point, which we denote by s. Let the circle Z be the section of the cone K by the plane V. Draw two tangents from the point s to the circle Z in the plane V. Denote the points of tangency by a and b. Each of the lines sa and sb is tangent to the cone K in one of these points. The lines ra and rb are generators of the cone. Therefore, each of the planes of the triangles asr and bsr contains a generator and a tangent of the cone K, and these lines intersect. Hence, these are tangent planes to the cone, and they meet along the line $sr = L$.

Lemma 8.3. *Any automorphism takes any C-object to a C-object and any P-object to a P-object of the same type. Any T-line is taken into a continuous line intersecting the interior of the I-cone (we do not claim here that this image is a line).*

Proof. By the definition of an automorphism, the image and the preimage of any I-cone is an I-cone. Since any bijection is strictly monotone with respect to the embedding of subsets, the boundary (the surface) of any I-cone is taken to the surface of the image. Every C-line is a line of tangency of two I-cones one of which is placed inside the other (tangency along a generator). This is an intersection of two surfaces of I-cones, which passes to a similar tangency, i.e., the image of any C-line is a C-line again. In this case, every C-plane is taken to a continuous set consisting of disjoint C-lines. Choose an arbitrary point r on an arbitrary C-plane B and draw a C-line L through r along which the plane B is tangent to the cone K. Each line of this kind of the C-plane is a tangent line of infinitely many I-cones whose vertices belong to L. Consider the set Q_L of all I-cones of this kind. Their union $K_L = \cup Q_L$ is a half-space bounded by the plane B. Under our automorphism, this union of cones is taken to a similar union, and the boundary of the half-space is taken to the boundary of the image. Hence, the image of any C-plane is a C-plane. Thus, any automorphism takes any C-object to a C-object of the same type.

By Lemma 8.2, the image of a P-line passing through the vertex of an I-cone K is the intersection of the images of two C-planes, and, as was proved above, this intersection is a P-line because it can be represented as the intersection of two C-planes external with respect to the I-cone given by the image of K. Since all lines on a P-plane are P-lines, it follows that any automorphism takes all lines on a P-plane into lines. Any three P-lines whose intersections define a triangle are taken by any automorphism to a similar pattern. By the continuity (Lemma 8.1) and the bijective property of any automorphism, the corresponding triangle defines a plane which is the image of the original plane. All lines in the image are P-lines, and hence the image is a P-plane. The interior of an I-cone passes under any automorphism to the interior of the target I-cone. Therefore, it follows from Lemma 8.1 that any T-line is taken to a continuous curve passing through the interior of the target I-cone.

Lemma 8.4. *The automorphisms are affine on the C-objects and the P-objects.*

Proof. For any P-plane, this was proved in the proof of Lemma 8.3. Since every P-line belongs to some P-plane, it follows that the automorphism is affine on the line. On any C-plane tangent to an I-cone (and hence not contained in the I-cone), all lines are either C-lines or P-lines. Each of these lines is taken by any automorphism to a line. Thus, the automorphism is affine on any C-plane. Since every C-line is contained to some C-plane, it follows that the mapping is affine on any C-line.

Lemma 8.5. *Any T-object is taken to a T-object of the same type under any automorphism u of a conic space.*

Proof. Consider an arbitrary T-plane P. By definition, there is an I-cone K restricted with respect to height by a circular section Z, and the intersection $P \cap K$ consists of two generatrices L_1 and L_2 and the vertex A. The intersection $L_3 = P \cap Z$ is a PP-line. By Lemma 8.4, uL_1 and uL_2 are generatrices of the I-cone uK which meet at the vertex uA of uK, and the image of the P-line uL_3 is a line and belongs to the plane uZ. Since the entire plane P can be represented as the union of C-lines parallel to L_1 and intersecting L_2 and L_3, it follows that uP is a T-plane. Thus, the image of an arbitrary T-plane is a T-plane. Every T-line is the intersection of two T-planes and meets the interior of any I-cone whose vertex belongs to the line. The property to enter the interior of any cone of this kind is an invariant of any automorphism. The intersection of two T-planes is taken to the intersection of their image planes. Hence, the image of an arbitrary T-line is a T-line.

End of the proof of Theorem 8.1. It follows from Lemmas 8.4 and 8.5 that, if the dimension of the space is equal to 3, then all automorphisms preserve the class of lines. This means that all the automorphisms are affine transformations of the vector space. Let $n > 3$. Note that the proof for $n = 3$ did not use any mapping of the conic space onto itself. Moreover, we have proved that a homeomorphism of any three-dimensional conic space is an affine mapping. Suppose that the desired assertion is proved for $n = m \leqslant 3$. Consider the case $n = m + 1$ and proceed by induction. Choose a basis: $\mathbb{R}^n = L\{e_0, e_1, \dots, e_m\}$, where e_0 is the axis of the I-cone. One can write out the algebraic sum $\mathbb{R}^n = L\{e_0, e_1, \dots, e_{m-1}\} \oplus L\{e_0, e_1, \dots, e_{m-2}, e_m\}$. Each of the summands is an m-dimensional conic space. Under any automorphism (or isomorphism) u, by the induction assumption, the images of these subspaces are subspaces of the same type, and the mappings are affine. The dimension of the space is preserved because the mapping u is bijective. If a line $l \subset \mathbb{R}^n$ is given, then we can always choose e_1 and e_2 in such a way that $l \subset L\{e_0, e_1, e_2\}$, and hence $l \subset L\{e_0, e_1, \dots, e_{m-1}\}$ (this is achieved, for instance, by a successive orthogonalization of the triple of vectors e_0, v, $w - v$, where w and v are any two distinct points on the line l). Therefore, the image of ul is a line. This completes the induction, and thus proves the theorem.

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